

SIGNAL-INVARIANCE S-Z TRANSFORMS FOR EFFECTIVE DIGITAL FILTER DESIGN

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ABSTRACT

A computer algorithm of the z-domain transfer function evaluation of a discrete-time (*DT*) system is presented. This system is designed from its continuous-time (*CT*) prototype using the signal invariance method. The impulse, step, and linear invariances are described. The set of two-graph modified nodal approach (2-graph *MNA*) equations, state-space equations or s-domain transfer function can be regarded as the model of analog counterpart. The procedure is based on the algorithm of numerical Laplace inversion.

KEY WORDS: Signal invariance, Transfer function, Numerical Laplace inversion.

1. INTRODUCTION

For digital filtering of biological, seismic and other signals, the requirements on the time-domain signal behaviour can be often primary. The method of signal invariance is a well-known procedure for designing *DT* systems from their *CT* prototypes [1]. The initially-at-rest responses to a given input signal for the designed *DT* system and for the original *CT* system are then equivalent. In this paper, the focus will be on the computer implementation of impulse, step, and linear invariances [1], [2].

The general design procedure can be described as follows:

1. Computation of the initially-at-rest response of the *CT* counterpart.
2. Sampling the response to obtain an equivalent initially-at-rest response of *DT* system.
3. Computation of the z-transforms of the sampled response and the sampled input signal.
4. Derivation of the system function as a ratio of the above z-transforms.

However, this procedure is not convenient for computer implementation. The realization of the first step means a

precise time analysis of the *CT* system. The algorithmization of the following items is difficult, especially for possible slowly converging or even diverging responses. In this procedure, there are accuracy limitations to deriving the system function in closed form. An efficient computer method is described in [3], but on the assumption that the state-space equations of the *CT* system are known. This approach can be also implemented if the design is based on transfer function $K(s)$ because the construction of state-space equations from $K(s)$ is trivial [4].

However, linear systems are often described by a set of differential-algebraic equations. The 2-graph *MNA* is widely used and popular in CAD simulators and design systems for *CT* applications. To utilise these design tools for designing *DT* applications, the 2-graph *MNA* model of *CT* counterpart is preferably considered.

The paper is organized as follows: The main idea of the approach is explained in Chapter 2. The algorithm is summarized in Chapter 3. In Chapter 4, we show one application example.

2. IMPULSE, STEP AND LINEAR INVARIANCES

Let us consider the 2-graph *MNA* equations of a linear system with one input signal w and one output signal y :

$$\mathbf{G}\mathbf{v}(t) + \mathbf{C} \frac{d}{dt} \mathbf{v}(t) = \mathbf{D}w(t), \quad y(t) = \mathbf{B}\mathbf{v}(t) \quad (1),(2)$$

where $\mathbf{v}$ is the $(nx1)$ vector of nodal variables, $\mathbf{B}$, $\mathbf{C}$ are the (nxn) matrices, and $\mathbf{D}$ and $\mathbf{B}$ are the $(nx1)$ and the $(1xn)$ vectors, respectively. At least one of matrices $\mathbf{G}$ and $\mathbf{C}$ is often singular.

In spite of this singularity, differential equation (1) has a solution in the following form. After discretizing the time axis into intervals $(kT, kT+T)$, $k = \dots -1, 0, 1, 2, \dots$, $T > 0$, the solution for the end of k -th interval is in the s-domain

$$\begin{aligned} (\mathbf{G} + s\mathbf{C})\mathbf{V}(s) &= \mathbf{C}\mathbf{v}(kT) + \mathbf{D}W(s) \Rightarrow \\ \mathbf{V}(s) &= (\mathbf{G} + s\mathbf{C})^{-1} \mathbf{C}\mathbf{v}(kT) + (\mathbf{G} + s\mathbf{C})^{-1} \mathbf{D}W(s), \end{aligned} \quad (3)$$

and in the time-domain

$$\mathbf{v}(kT+T) = \mathbf{g}^*(T)\mathbf{v}(kT) + \int_0^T \mathbf{g}(\xi)w(kT+T-\xi)d\xi, \quad (4)$$

where

$$\mathbf{g}^*(T) = L^{-1}\{(\mathbf{G} + s\mathbf{C})^{-1}\}\mathbf{C}|_{t=T} \quad (5)$$

is the (nxn) matrix of zero-input response for time $t=T$, and

$$\mathbf{g}(T) = L^{-1}\{(\mathbf{G} + s\mathbf{C})^{-1}\}\mathbf{D}|_{t=\xi} \quad (6)$$

is the $(nx1)$ vector of impulse response.

As mentioned in (5) and (6), these characteristics can be obtained by using the algorithm of numerical Laplace inversion (*NLI*). This algorithm is described in [4] and [5] and will not be explained here.

Approximating input signal $w(t)$ during a given time interval by various terms, equation (4) can be converted into a difference equation for the *DT* system. For the approximations given below, the difference equation is in the general form

$$\mathbf{v}(kT+T) = \mathbf{g}^*(T)\mathbf{v}(kT) + \mathbf{G}w(kT) + \mathbf{H}w(kT+T). \quad (7)$$

It is known that replacing the signal $w(t)$, $t \in (kT, kT+T)$ by Dirac impulse, constant step or linear signal in accordance with Table 1, the corresponding difference equation (7) describes a *DT* system, designed from *CT* model (1) using impulse invariance, step invariance and linear invariance, respectively. The $\mathbf{G}$ and $\mathbf{H}$ vectors for all the invariances are in Table 1.

Numerical evaluation of these vectors can be performed by means of *NLI*: $\mathbf{g}(0^+)$ by two-step solution with forward and backward integration, and $\mathbf{h}(T)$ and $\mathbf{q}(T)$ by the rule of Laplace transform of integral [5].

Applying the z-transform, equation (7) can be derived as follows:

$$[z\mathbf{E} - \mathbf{g}^*(T)]\mathbf{V}(z) = (\mathbf{G} + \mathbf{H}z)\mathbf{W}(z), \quad (8)$$

where $\mathbf{E}$ is the identity matrix.

It should be noted that the poles of the designed *DT* system are the eigenvalues of matrix $\mathbf{g}^*(T)$. Taking into account output equation (2), the coefficients of the z-

domain transfer function $K(z) = y(z)/w(z)$ can be obtained by using the *QZ* algorithm or the *Faddejev* algorithm or by other known procedures.

Obviously, the above algorithm provides correct results also on the assumption that the *CT* system is described by state-space equations instead of equations (1), because state-space equations are the simplified cases of (1) for regular matrix $\mathbf{C}=\mathbf{E}$.

3. ALGORITHM

Let us summarize the algorithm. The input data are 2-graph *MNA* models (1) and (2) of the *CT* prototype, the sampling frequency of *DT* system $f_s = 1/T$, and the chosen type of invariance (impulse, step, or linear). The output data are the zeros and poles (or numerator and denominator coefficients) of the z-domain transfer function of the *DT IIR* (infinite impulse response) system.

Algorithm:

1. We compute $\mathbf{g}(0^+)$, $\mathbf{g}^*(T)$, $\mathbf{h}(T)$ and $\mathbf{q}(T)$ numerically using the algorithms of numerical Laplace inversion as described in [5].
2. We compute the $\mathbf{G}$ and $\mathbf{H}$ vectors according to Table 1.
3. We compute the z-domain poles of the *DT* system, which are for all the invariances the same, as the eigenvalues of matrix $\mathbf{g}^*(T)$.
4. We compute the z-domain zeros of the *DT* system according to equation (8), taking into account output equation (2).
5. If necessary, we obtain the coefficients of z-domain transfer function after the expansion of root factors.

4. ILLUSTRATION EXAMPLE

To verify the above approach, a computer program has been constructed. As an example, the 9-th order Cauchy approximation filter has been designed. The *CT* prototype has been designed using the *NAF* system [6]. The resulting s-domain coefficients are in Table 2. The cut-off

invariance	impulse	step	linear
approximation of $w(t)$ $t \in (kT, kT+T)$	$Tw(kT+T)\delta(t-kT-T)$	$w(kT)$	$w(kT) + a.(t-kT)$ $a = [w(kT+T) - w(kT)]/T$
vector $\mathbf{G}$	$\mathbf{0}$	$\mathbf{h}(T) = \int_0^T \mathbf{g}(\xi)d\xi$	$\mathbf{h}(T) - \mathbf{q}(T)$
vector $\mathbf{H}$	$\mathbf{g}(0^+)T$	$\mathbf{0}$	$\mathbf{q}(T) = \frac{1}{T} \int_0^T \mathbf{h}(\xi)d\xi$

Table 1. Various types of invariance and the corresponding vectors $\mathbf{G}$ and $\mathbf{H}$ from Eq. (8).

frequency is normalized to 1 Hz. The calculated z-domain coefficients of *DT* filters are in Table 3. A sampling frequency of 10 Hz has been chosen. Fig. 1 shows the resulting frequency responses.

$K(s) = \frac{a_0 + a_1s + a_2s^2 + \dots + a_ns^n}{b_0 + b_1s + b_2s^2 + \dots + b_ns^n}$		
i	a_i	b_i
0	2.63595E5	2.63595E5
1	0	3.53177E5
2	1.39877E4	1.16327E5
3	0	6.97929E4
4	2.59884E2	1.04413E4
5	0	4.39004E3
6	1.93506E0	3.33814E2
7	0	1.11657E2
8	4.50853E-3	3.54216E0
9	0	1.00000E0

Table 2. s-domain coefficients of *CT* prototype.

$K(z) = \frac{c_0 + c_1z^{-1} + c_2z^{-2} + \dots + c_nz^{-n}}{d_0 + d_1z^{-1} + d_2z^{-2} + \dots + d_nz^{-n}}$				
i	d_i	c_i		
	all invar.	impulse	step	linear
0	1.0000E0	4.5085E-4	0	2.5795E-4
1	-7.6423E0	-2.4554E-3	6.0594E-4	-8.6042E-4
2	2.6866E1	7.5579E-3	-2.7696E-3	1.6228E-3
3	-5.6871E1	-1.4964E-2	6.9021E-3	-1.8044E-3
4	7.9764E1	2.0491E-2	-1.1167E-2	1.2458E-3
5	-7.6797E1	-1.9396E-2	1.2959E-2	-8.9829E-5
6	5.0738E1	1.2516E-2	-1.0868E-2	-8.1187E-4
7	-2.2184E1	-5.0355E-3	6.5688E-3	1.0926E-3
8	5.8277E0	1.0388E-3	-2.5971E-3	-6.7472E-4
9	-7.017E-1	0	5.6697E-4	2.2428E-4

Table 3. z-domain coefficients of designed *DT* systems.

As shown in Fig. 1, the linear invariance guarantees the best approximation of the *DT* frequency response to the original response of *CT* prototype.

5. CONCLUSION

The algorithms described start from the two-graph modified nodal description of a linear *CT* system and transfer it into the z-domain transfer function of *DT* system according to the impulse, step, and linear invariances. The procedures are based on analytical preprocessing of the final z-domain transfer function. That is why they do not use ill-conditioned algorithms such as partial fraction expansion, which are often utilized in these

cases. Owing to the algorithm of numerical Laplace inversion, this method provides accurate results.

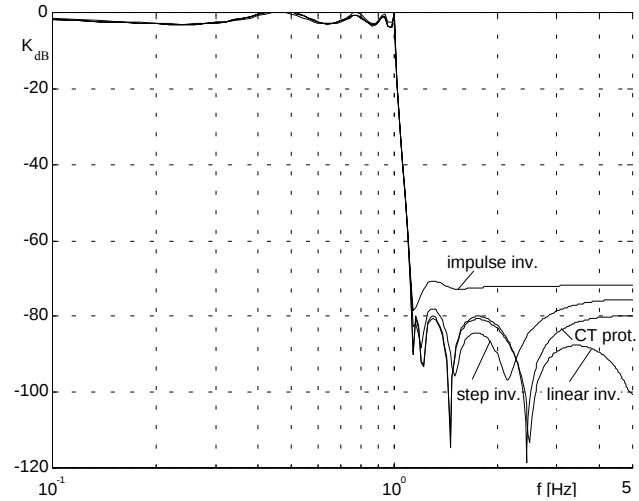


Fig.1. Frequency responses of *CT* prototype and *DT* systems according to the impulse, step and linear invariance.

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