GENERALIZED PASCAL MATRIX OF FIRST ORDER S-Z TRANSFORMS

Viera Biolková) and Dalibor Biolek**)*

*) Institute of Radioelectronics, BUT, Purkyòova 118, 612 00 Brno, Czech Republic
Tel. (+420) 5 41149 152, fax (+420) 5 41149 244, email: biolkova@urel.fee.vutbr.cz

**) Institute of Telecommunications, BUT, Purkyòova 118, 612 00 Brno, Czech Republic
Tel. (+420) 5 41149 190, fax (+420) 5 41149 192, email: biolek@cs.vabo.cz

ABSTRACT

The generalized Pascal matrix is introduced in this contribution. Using this matrix, the coefficients of transfer functions of the continuous-time and discrete-time linear circuits can be recalculated on the assumption that both circuits are connected by a general first-order *s-z* transformation.

1. INTRODUCTION

The so-called *s-z* transformations are often used for modeling interconnections between the continuous-time (CT) and discrete-time (DT) systems. The bilinear (BL), backward-difference (BD), forward difference (FD) and lossless discrete integration (LDI) belong to the well known transformations [1]. The BL transformation is commonly used for the DT filters from the CT prototypes. The BD, FD and LDI transformations play an important role in the synthesis and modeling of switched-capacitor and switched-current filters. All these transformations can be used for the discretization of CT systems, for instance for the purpose of their numerical analysis.

In addition to the above *s-z* transformations, a set of so-called parametric transformations have been introduced [2], [3]. Their properties can be modified by an auxiliary parameter. To change it, we can pass, for example, from the BL to the BD transformation.

All known *s-z* transformations can be divided into first-order and higher-order ones [3]. The first-order transformations ensure that the order of the transformed system will be the same as the order of the original system. From the transformations mentioned above, the LDI does not fulfill this requirement.

For such transformations between the continuous- and discrete-time systems which are connected by a concrete *s-z* transformation, the mutual recalculation of the transfer function coefficients is necessary. This recalculation can be done by an effective algorithm which is advantageous especially for the design of high-order systems. In case of the often used bilinear transformation, the algorithm is well known as so-called Pascal matrix [4].

In this article, we derive a matrix algorithm for recalculating coefficients of the transfer functions for general first order *s-z* transformation. This algorithm will include the method of Pascal matrix as a special case. Then we concretize the algorithm for the BL, BD, FD transformations, and for one of the parametric transformations.

2. FIRST-ORDER S-Z TRANSFORMS

The well-known first-order transformations are as follows:

<u>Bilinear (BL)</u>

$$s = a\frac{z-1}{z+1}, \text{ or } z = \frac{a+s}{a-s}, \tag{1}$$

$$a = \frac{2}{T} = 2F \,. \tag{2}$$

<u>Backward-Difference (BD)</u>

$$s = b\left(1-z^{-1}\right) = b\frac{z-1}{z}, \text{ or } z = \frac{b}{b-s}, \tag{3}$$

$$b = \frac{1}{T} = F \,. \tag{4}$$

<u>Forward-Difference (FD)</u>

$$s = b(z-1), \text{ or } z = 1+\frac{s}{b} \,. \tag{5}$$

<u>Parametric BD-BL transformation</u> [3]

$$s = \frac{1+r}{T}\frac{z-1}{z+r}, \text{ or } z = -\frac{1+r+srT}{sT-1-r} \,. \tag{6}$$

To change parameter r from 0 to 1, we pass from the BD to the BL transformation.

The first-order *s-z* transformation can be defined in the following way:

$$s = \frac{m_1+m_2 z}{m_3+m_4 z}, \text{ or } z = \frac{m_1-m_3 s}{-m_2+m_4 s}, \tag{7}$$

where m_1, m_2, m_3 and m_4 are real constants.

For the practical utilization of *s-z* transformation, the *s*-plane zero-point has to be transformed into the point [1,0] of the complex plane *z*:

$$s = 0 \leftrightarrow z = 1.$$

For this reason, the constants m_1, m_2, m_3 and m_4 in (7) could not be chosen arbitrarily, because the following relations must be fulfilled:

m_1 = -m_2
$m_3 \neq$ -m_4.

Let us respect these relations using the following form of equation (7):

$$s = u\frac{1-z}{-v+wz}\ \text{, or}\ z = \frac{u+vs}{u+ws}\ , \tag{8}$$

where u, $v \neq w$ are real constants. Choosing a proper set of constants leads to all the known transformations such as BL, BD, FD, and all parametric first-order ones.

3. GENERALIZED PASCAL MATRIX

Consider a DT linear system with the transfer function

$$K_{DT}(z) = \frac{\sum_{k=0}^{N} a_k z^{-k}}{\sum_{k=0}^{N} b_k z^{-k}} = \frac{\sum_{k=0}^{N} a_k z^{N-k}}{\sum_{k=0}^{N} b_k z^{N-k}}\ , \tag{9}$$

where N is the system order. The sampling frequency will be F and the sampling period T (see equation 2).

Consider the CT linear system with the transfer function

$$K_{CT}(s) = \frac{\sum_{k=0}^{N} c_k s^k}{\sum_{k=0}^{N} d_k s^k}\ . \tag{10}$$

Let us assume that the DT system has been designed from the CT prototype using the s-z transformation (8). Then an unambiguous relation between the s- and z- domain coefficients has to exist.

Applying s-z transformation (8) to the z-domain transfer function (9) yields

$$K_{CT}(s) = \frac{\sum_{k=0}^{N} a_k \left(\frac{u+vs}{u+ws}\right)^{N-k}}{\sum_{k=0}^{N} b_k \left(\frac{u+vs}{u+ws}\right)^{N-k}} = \frac{\sum_{k=0}^{N} a_k (u+vs)^{N-k}(u+ws)^k}{\sum_{k=0}^{N} b_k (u+vs)^{N-k}(u+ws)^k} \tag{11}$$

Comparing (11) and (10), the coefficients c_k (d_k) can be determined from the set of coefficients a_k (b_k). Because of the formal identity of the numerator and denominator in (11), it is sufficient to derive transform relations only for the numerator.

Using the binomical rule and making the necessary simplifications, we obtain the following equations:

$$c_k = u^{N-k}\sum_{i=0}^{N} a_i M_{k,i}\ , \quad k = 0,1,\dots,N\ , \tag{12}$$

$$M_{k,i} = \sum_{n=0}^{k}\binom{N-i}{n}\binom{i}{k-n}v^n w^{k-n} =$$

$$= \binom{i}{k}w^k + \sum_{n=1}^{k-1}\binom{N-i}{n}\binom{i}{k-n}v^n w^{k-n} + \binom{N-i}{k}v^k\ ,\quad k,i = 0,1,\dots N \tag{13}$$

Equations (12) and (13) represent the algorithm how to compute the numerator coefficients of DT circuits from the numerator coefficients of CT circuits. In the limiting cases, the evaluation of the binomical coefficients is performed according to the following rules:

$$\binom{n}{k} = \begin{matrix} 0 \text{ for } k < 0 \text{ and } n < k \\ 1 \text{ for } k = 0 \end{matrix} \tag{14}$$

Let us arrange the coefficients a_k of the DT circuit and the coefficients c_k of CT circuit to the vectors

$$\mathbf{A} = [a_0\ a_1\ a_2\ \cdots a_N]^T,\quad \mathbf{C} = [c_0\ c_1\ c_2\ \cdots c_N]^T\ , \tag{15}$$

and define auxiliary vector

$$\tilde{\mathbf{C}} = [c_0 u^{-N}\ c_1 u^{1-N}\ c_2 u^{2-N}\ \cdots c_N u^{N-N}]^T\ . \tag{16}$$

Then equation (12) can be rewritten in the matrix form

$$\tilde{\mathbf{C}} = \mathbf{MA}\ , \tag{17}$$

$$\mathbf{M} = \begin{bmatrix} 1 & 1\ \ 1\ \ 1\ \cdots & 1 \\ \binom{N}{1}v^1 & \cdots\ \cdots\ \cdots\ \cdots & \binom{N}{1}w^1 \\ \binom{N}{2}v^2 & \cdots\ \cdots\ \cdots\ \cdots & \binom{N}{2}w^2 \\ \binom{N}{3}v^3 & \cdots\ \cdots\ \cdots\ \cdots & \binom{N}{3}w^3 \\ \vdots & \cdots\ \cdots\ \cdots\ \cdots & \vdots \\ \binom{N}{N}v^N & \cdots\ \cdots\ \cdots\ \cdots & \binom{N}{N}w^N \end{bmatrix} \tag{18}$$

M is the ($N+1$) x ($N+1$) matrix. Its components are defined by equation (13). Let us call it the *generalized Pascal matrix*.

The first matrix row contains only the ones:

$$M_{0,i} = \sum_{n=0}^{0}\binom{N-i}{n}\binom{i}{-n}v^n w^{-n} = 1,\ \ i = 0,1,\dots N\ .$$

The first (last) column contains binomical coefficients $\binom{N}{k}$ multiplied by the powers of coefficient v (w):

$$M_{k,0} = \sum_{n=0}^{k}\binom{N}{n}\binom{0}{k-n}v^n w^{k-n} = \binom{N}{k}v^k\ ,$$

$$M_{k,N} = \sum_{n=0}^{k}\binom{0}{n}\binom{N}{k-n}v^n w^{k-n} = \binom{N}{k}w^k\ , k = 0,1,\dots N\ .$$

The other elements of the generalized Pascal matrix are determined by more complicated relations, see (13).

BL	BD	FD	parametric BL-BD
$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 5 & 3 & 1 & -1 & -3 & -5 \\ 10 & 2 & -2 & -2 & 2 & 10 \\ 10 & -2 & -2 & 2 & 2 & -10 \\ 5 & -3 & 1 & 1 & -3 & 5 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 & -4 & -5 \\ 0 & 0 & 1 & 3 & 6 & 10 \\ 0 & 0 & 0 & -1 & -4 & -10 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 \\ 10 & 6 & 3 & 1 & 0 & 0 \\ 10 & 4 & 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 5r & -1+4r & -2+3r & -3+2r & -4+r & -5 \\ 10r^2 & -4r+6r^2 & 1-6r+3r^2 & 3-6r+r^2 & 6-4r & 10 \\ 10r^3 & -6r^2+4r^3 & 3r-6r^2+r^3 & -1+6r-3r^2 & -4+6r & -10 \\ 5r^4 & -4r^3+r^4 & 3r^2-2r^3 & -2r+3r^2 & 1-4r & 5 \\ r^5 & -r^4 & r^3 & -r^2 & r & -1 \end{bmatrix}$

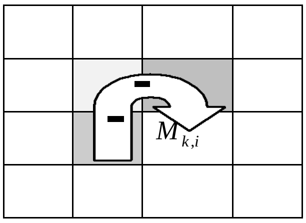
Fig.1. An example of generalized Pascal matrices for $N = 5$.

4. INSTANTIATION FOR KNOWN S-Z TRANSFORMATIONS

Bilinear transformation: $u = a, v = 1, w = -1$.

An arbitrary element out of the first row and column can be determined from the neighbouring elements from the recurrent formula

$$M_{k,i} = M_{k,i-1} - M_{k-1,i-1} - M_{k-1,i}. \qquad (19)$$



Because we know marginal elements (18), applying (19) subsequently leads to the simple generation of the whole matrix.

The inverse Pascal matrix is given by a simple equation [4]

$$\mathbf{M}^{-1} = 2^{-N}\mathbf{M}. \qquad (20)$$

Transformation BD: $u = b, v = 0, w = -1$.

The matrix elements are given by the simple equation

$$M_{k,i} = \binom{i}{k}(-1)^k \text{ (see example in Fig. 1).}$$

The recurrent formula is now

$$M_{k,i} = M_{k,i-1} - M_{k-1,i-1} \qquad (21)$$

For BD transformation, the Pascal matrix is the same as its inversion:

$$\mathbf{M}^{-1} = \mathbf{M}. \qquad (22)$$

Transformation FD: $u = b, v = 1, w = 0$.

$$M_{k,i} = \binom{N-i}{k} \text{ (see example in Fig. 1).}$$

$$M_{k,i} = M_{k,i+1} + M_{k-1,i+1}. \qquad (23)$$

To perform Pascal matrix inversion, it is sufficient to interchange sequence of rows and columns of the original matrix **M**, and to multiple each element by the number $(-1)^{k+l+1}$, where k,l are the row and column positions.

Parametric BD-BL transformation:

$$u = \frac{1+r}{T}, v = r, w = -1.$$

$$M_{k,i} = \sum_{n=0}^{k} \binom{N-i}{n}\binom{i}{k-n}(-1)^{k-n} r^n \text{ (see example in Fig. 1).}$$

The recurrent formula and the closed-form inverse Pascal matrix are now rather complicated.

5. CONCLUSION

Equation (13) offers directions how to compile a generalized Pascal matrix of an arbitrary first-order *s-z* transformation which is defined by equation (8). Equation (12) shows how to use it for recalculating coefficients of CT and DT systems.

There exists one degree of freedom to assign constants u, v and w to a given *s-z* transformation. From this point of view, definitions (12) and (13) of the Pascal matrix are ambiguous. Then it could be advantageous to choose constants v, w (are the component parts of the matrix) and u (is not the matrix part) in such a way as to yield the simplest Pascal matrix possible.

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6. REFERENCES

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