Modeling of Periodically Switched Networks by Mixed s-z Description

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***Abstract*—A new description of linear periodically time-varying systems is proposed. To describe simultaneously the continuous and discrete time nature of the system, the generalized transfer function of the Laplace and discrete operator is introduced. This mixed description includes the transfer functions of continuous time and discrete time systems as limiting cases. Special attention is paid to two-phase switched circuits and results of computer simulations are presented.**

***Index Terms*— Generalized transfer function, modified nodal analysis, switched network.**

I. Introduction

MANY electronic systems can be modeled with periodically varying parameters and time-varying linear systems can be modeled by nonstationary transfer functions $K(s,t)$ [1]. Because the parameters vary periodically, the system response contains both continuous and discrete time parts and both aspects can be expressed by a generalized transfer function (GTF) $K(s,z)$ [2]–[4]. Using the equivalent signal theory by Tsividis [5], the frequency response is obtained by double substitution $s = j\omega, z = \exp(j\omega T)$, where T is the period. Such a hybrid description includes also the continuous-time and discrete-time transfer functions as limiting cases. Dynamic properties of time-varying systems can be described by pole locations of the GTF in the s- and z- domains [6].

This contribution explains the main ideas of linear time-varying system modeling using GTF's. We first introduce mathematical derivations, with some results from [3]–[8]. Next, we consider networks with discontinuous parameters, and finally we present simulation results.

II. What is the GTF?

Consider a time-varying system with input $w(t)$, output $v(t)$, and a period of time changes of parameters, T. Let us sample the output signal at instances $kT + \varepsilon T$ for a fixed value $\varepsilon \in \langle 0, 1\rangle$. A unique continuous-time signal $v_\varepsilon^e(t)$, also called the equivalent signal, can be constructed to have the following properties:

1) represent interpolation of the sequence $\{v(kT + \varepsilon T)\}_{k=\cdots-1,0,1\cdots}$;
2) have spectral components within the spectrum of the input signal $w(t)$.

The first property indicates that the equivalent and output signals have common points at the sampling instances $kT+\varepsilon T$. Depending on ε, various equivalent signals can represent the output signals and create a set of envelopes.

The second property enables us to define the *generalized transfer function* (GTF) as a ratio of Fourier/Laplace transformations of equivalent output and input signals. Depending on the parameter ε, the various GTF's can represent single network behavior. It can be proved that the signal $v(t)$ is identical with all its equivalent signals $v_\varepsilon^e(t)$, if the Nyquist condition

$$f_{\max} < \frac{1}{2T} \tag{1}$$

is satisfied. Here $f_{\max}$ is the maximum frequency of nonzero spectral components of the signal $v(t)$.

To explain the idea, consider the simplified responses of the sample and hold (SH) circuit in Fig. 1. Due to the inertia of the circuit in the time interval $\langle kT, kT + \varepsilon T\rangle$, a dynamic error $\Delta_\varepsilon = v(kT + \varepsilon T) - w(kT + \varepsilon T)$ occurs in the sample definition. This error can be described by the GTF theory. The equivalent signal $v_\varepsilon^e(t)$ is defined for the sequence $\{v(kT + \varepsilon T)\}$. In case of precise sample selection without errors, this signal and the input signal are identical. The error can be investigated by comparison of these signals in time and frequency domain. The GTF of a SH circuit which models the error can be defined as

$$K = \frac{L[v_\varepsilon^e(t)]}{L[w_\varepsilon^e(t)]} = \frac{L[v_\varepsilon^e(t)]}{L[w(t)]} \tag{2}$$

with $L[\ ]$ denoting the Laplace transform operator. The last equality is true if the signal $w(t)$ satisfies the Nyquist condition (1).

Other errors occurring during the SH process can also be modeled by the GTF. For example, the voltage-drop error during time interval T_p can be investigated by comparing equivalent signals $v_\varepsilon^e(t)$ and $v_{T_p/T}^e(t)$.

Some of the above problems have been solved in [3]. As described below, the GTF is a function of both s and z operators. Substitution of $s = j\omega$ and $z = e^{j\omega T}$ leads to the frequency response.

Let us examine the time response of the two-phase second-order SC filter in Fig. 2. Due to switching, the output signal contains high frequency spectral components. Two continuous-time equivalent signals $v_{5,1}^e$ and $v_{5,2}^e$ create certain envelopes of impulse signals. Comparing equivalent signals with the

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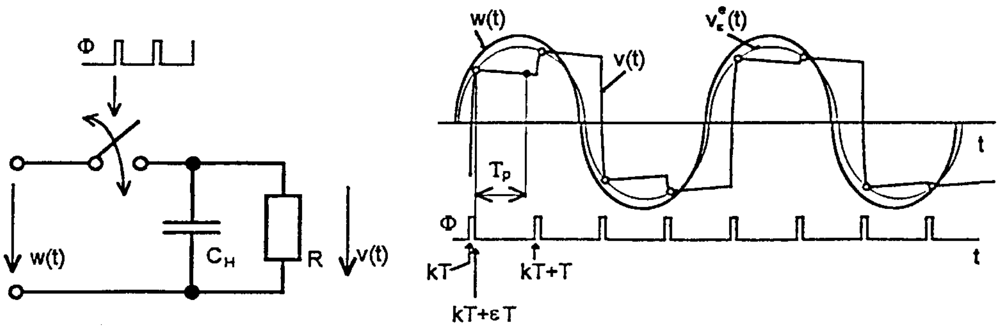
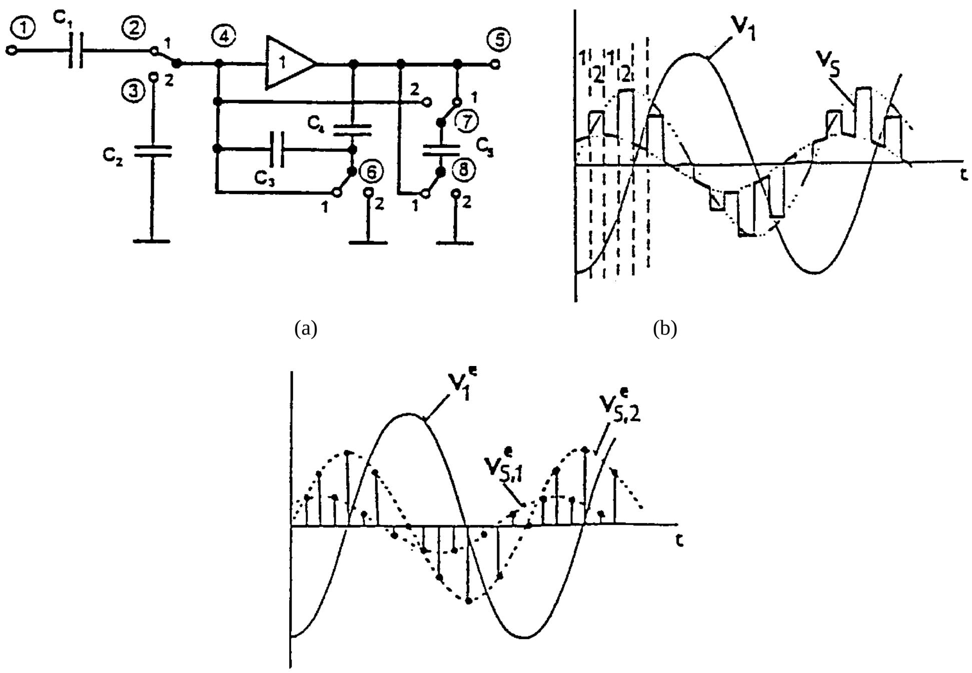


Fig. 1. Simplified model of a sample-and-hold circuit and corresponding responses, including the equivalent signal $v_\varepsilon^e(t)$.



(a) (b)

(c)

Fig. 2. (a) Switched capacitor filter. (b) Its actual signals. (c) Its equivalent signals.

input signal $v_1 = v_1^e$ for various frequencies provides information on the transient properties in both switching phases.

The system with time-varying parameters can be described by the relation of Fourier (Laplace) transforms of certain equivalent output and input signals. This way a set of GTF's can be obtained for a single network, taking into account various types of output and input regimes. The GTF's can be defined as

$$K_1 = \frac{L[v_{5,1}^e]}{L[v_1^e]} \quad K_2 = \frac{L[v_{5,2}^e]}{L[v_1^e]}$$

In the following we present mathematical description of GTF's.

III. Mixed s-z Description of Periodically Varying Systems

Consider a linear time-varying system with period T, and a single input. The modified nodal approach [2] leads to the system equation

$$\boldsymbol{G}(t)\boldsymbol{v}(t) + \frac{d}{dt}\boldsymbol{C}(t)\boldsymbol{v}(t) = \boldsymbol{D}(t)w(t) \tag{3}$$

where $\boldsymbol{G}$ is the conductance matrix, $\boldsymbol{C}$ is the reactance matrix, $\boldsymbol{D}$ is the incidence vector, $\boldsymbol{v}$ the solution vector and w the input signal. Solving (3) for $t \in (kT+\varepsilon T - T, kT+\varepsilon T\rangle, k = \cdots, -1, 0, 1, \cdots$ yields

$$\begin{aligned}\boldsymbol{v}(t) = &\boldsymbol{g}^*(t, kT+\varepsilon T - T)\boldsymbol{v}(kT+\varepsilon T - T)\\ &+ \int_{kT+\varepsilon T - T}^{t} \boldsymbol{g}(t,\xi)w(\xi)\,d\xi\end{aligned} \tag{4}$$

where $\boldsymbol{g}^*(t, kT+\varepsilon T - T)$ is a two-dimensional initial condition matrix and $\boldsymbol{g}(t,\xi)$ is the impulse response matrix. At instances $t = kT + \varepsilon T$, (4) can be rewritten in the form

$$\begin{aligned}&\boldsymbol{v}(kT+\varepsilon T)\\ &\quad = \boldsymbol{g}^*(kT+\varepsilon T, kT+\varepsilon T - T)\boldsymbol{v}(kT+\varepsilon T - T)\\ &\qquad + \int_0^T \boldsymbol{g}_x(kT+\varepsilon T, \xi)w(kT+\varepsilon T - \xi)\,d\xi.\end{aligned} \tag{5}$$

Here $\boldsymbol{g}_x(kT+\varepsilon T,\xi) = \boldsymbol{g}(kT+\varepsilon T, kT+\varepsilon T-\xi)$ is the impulse response at time $kT+\varepsilon T$, on the assumption that a Dirac impulse has appeared at time delay ξ, before the time of observation at $kT+\varepsilon T$.

Due to the periodicity of system parameters, (5) can be simplified to the form

$$\begin{aligned}\boldsymbol{v}(kT+\varepsilon T) =& \boldsymbol{g}^*(\varepsilon T, \varepsilon T - T)\boldsymbol{v}(kT+\varepsilon T - T)\\ &+ \int_0^T \boldsymbol{g}_x(\varepsilon T,\xi) w(kT+\varepsilon T-\xi)\, d\xi. \quad (6)\end{aligned}$$

This is a recurrent formula of periodically varying systems. It can be solved using the idea of equivalent signals. Consider the equivalent signal vector

$$\boldsymbol{v}_\varepsilon^e(t)|_{t=kT+\varepsilon T} = \boldsymbol{v}(kT+\varepsilon T). \quad (7)$$

Replacing the discrete time variable by continuous time using the substitution

$$kT+\varepsilon T \to t \quad (8)$$

(6) can be rewritten for the equivalent signal

$$\begin{aligned}\boldsymbol{v}_\varepsilon^e(t) =& \boldsymbol{g}^*(\varepsilon T, \varepsilon T - T)\boldsymbol{v}_\varepsilon^e(t-T)\\ &+ \int_0^T \boldsymbol{g}_x(\varepsilon T,\xi) w(t-\xi)\, d\xi. \quad (9)\end{aligned}$$

This can be verified by comparing (7) and (9) with the properties of the equivalent signal given by points 1 and 2 in Section II. Equation (7) confirms point 1, that the equivalent and output signals have common points at the sampling instances $kT+\varepsilon T$. Since the equivalent and input signals in (9) are coupled by a linear equation, their frequency ranges are identical and point 2 is satisfied.

Applying Laplace transform to (9) yields

$$\begin{aligned}\boldsymbol{V}_\varepsilon^e(s) =& \boldsymbol{g}^*(\varepsilon T, \varepsilon T - T)\boldsymbol{V}_\varepsilon^e(s)e^{-sT}\\ &+ \int_0^\infty \int_0^T \boldsymbol{g}_x(\varepsilon T,\xi) w(t-\xi)e^{-st}\, d\xi\, dt\end{aligned}$$

After some rearrangements

$$\boldsymbol{V}_\varepsilon^e(s,z) = \boldsymbol{K}_\varepsilon(s,z)|_{z=\exp(sT)} W(s)$$

where $\boldsymbol{K}_\varepsilon(s,z)$ is the generalized transfer function (GTF) of the time-varying system

$$\boldsymbol{K}_\varepsilon(s,z) = [\boldsymbol{E} - \boldsymbol{g}^*(T+\varepsilon T, \varepsilon T)z^{-1}]^{-1}\boldsymbol{G}_\varepsilon(s) \quad (10)$$

and

$$\boldsymbol{G}_\varepsilon(s) = \int_0^T \boldsymbol{g}_x(\varepsilon T,\xi)e^{-s\xi}\, d\xi \quad (11)$$

is the short-time Laplace transform of the impulse response. $\boldsymbol{E}$ denotes the identity matrix. The GTF in (10) consists of two multiplicative parts, one a function of the z-operator, the other a function of the s-operator. For additional details concerning interpretation of GTF, see [3]–[5]. Dynamic properties of a linear time-varying system can be investigated by considering the s and z domain pole locations. The z-domain poles are decisive for system stability. They are eigenvalues of the matrix $\boldsymbol{g}^*(T+\varepsilon T, \varepsilon T)$. The s domain poles determine the transient process caused by the variation of system parameters.

Computation of the matrices $\boldsymbol{g}^*$ and $\boldsymbol{g}$ from the modified nodal description is the fundamental problem. In this contribution only modeling of systems with discontinuous parameters will be discussed. Switched capacitor networks, general switched filters and sample-hold circuits are typical applications.

IV. Systems with Discontinuous Parameters

Let the period T be divided into N phases, each of length T_i, and let the system be stationary in each phase. Also define $\sigma_k = \Sigma_{i=1}^k T_i$. In the ith phase, the equations in the s-domain have the form [4]

$$\begin{aligned}(\boldsymbol{G}_1 + s\boldsymbol{C}_1)\boldsymbol{V}_1(s) =& \boldsymbol{C}_{1,N}\boldsymbol{v}_N(kT^-) + \boldsymbol{D}_1 w_1(s)\\ (\boldsymbol{G}_i + s\boldsymbol{C}_i)\boldsymbol{V}_i(s) =& \boldsymbol{C}_{i,i-1}\boldsymbol{v}_{i-1}(kT+\sigma_{i-1}^-) + \boldsymbol{D}_i w_i(s)\\ & i = 2,3,\cdots N \quad (12)\end{aligned}$$

with $\boldsymbol{C}_{i,i-1}$ denoting the matrices of the phase-to-phase energy transfer. Application of the equivalent signal theory leads to a large system of equations (see (13) at the bottom of the page) where

$$\begin{aligned}\boldsymbol{g}_i(t) =& L^{-1}[\boldsymbol{Y}_i^{-1}(s)]\boldsymbol{D}_i\\ \boldsymbol{g}_i^* =& L^{-1}[\boldsymbol{Y}_i^{-1}(s)]\boldsymbol{C}_{i,i-1}\\ \boldsymbol{G}_i(s) =& \int_0^{T_i} \boldsymbol{g}_i(\xi)e^{-s\xi}\, d\xi. \quad (14)\end{aligned}$$

Equation (13) must be solved for the unknown vectors $\boldsymbol{V}_i$, to obtain the generalized transfer function. Analytical solution is

$$\boldsymbol{V}_N = \sum_{m=1}^N \boldsymbol{K}_m W_m \quad (15)$$

where

$$\begin{aligned}\boldsymbol{K}_m(s,z) =& \left[\boldsymbol{E} - z^{-1}\prod_{i=1}^{N-m-1} \boldsymbol{g}_{N-i}^*\right]^{-1} \prod_{i=0}^{N-m-1} \boldsymbol{g}_{N-i}^*\\ &\cdot e^{-sT_{N-i}}\boldsymbol{G}_m(s) \quad (16)\end{aligned}$$

Before going into computer implementation, we present one example how to generate the GTF, and explain some aspects

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -\boldsymbol{g}_1^* e^{-sT_1} \\ -\boldsymbol{g}_2^* e^{-sT_2} & 1 & 0 & 0 & 0 \\ 0 & -\boldsymbol{g}_3^* e^{-sT_3} & 1 & 0 & 0 \\ \cdots & & \cdots & & \\ 0 & 0 & 0 & -\boldsymbol{g}_N^* e^{-sT_N} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_1^e \\ \boldsymbol{V}_2^e \\ V_3^e \\ \\ \boldsymbol{V}_n^e \end{bmatrix} = \begin{bmatrix} \boldsymbol{G}_1(s)\boldsymbol{W}_1(s) \\ \boldsymbol{G}_2(s)\boldsymbol{W}_2(s) \\ \boldsymbol{G}_3(s)\boldsymbol{W}_3(s) \\ \cdots \\ \boldsymbol{G}_N(s)\boldsymbol{W}_N(s) \end{bmatrix}. \quad (13)$$

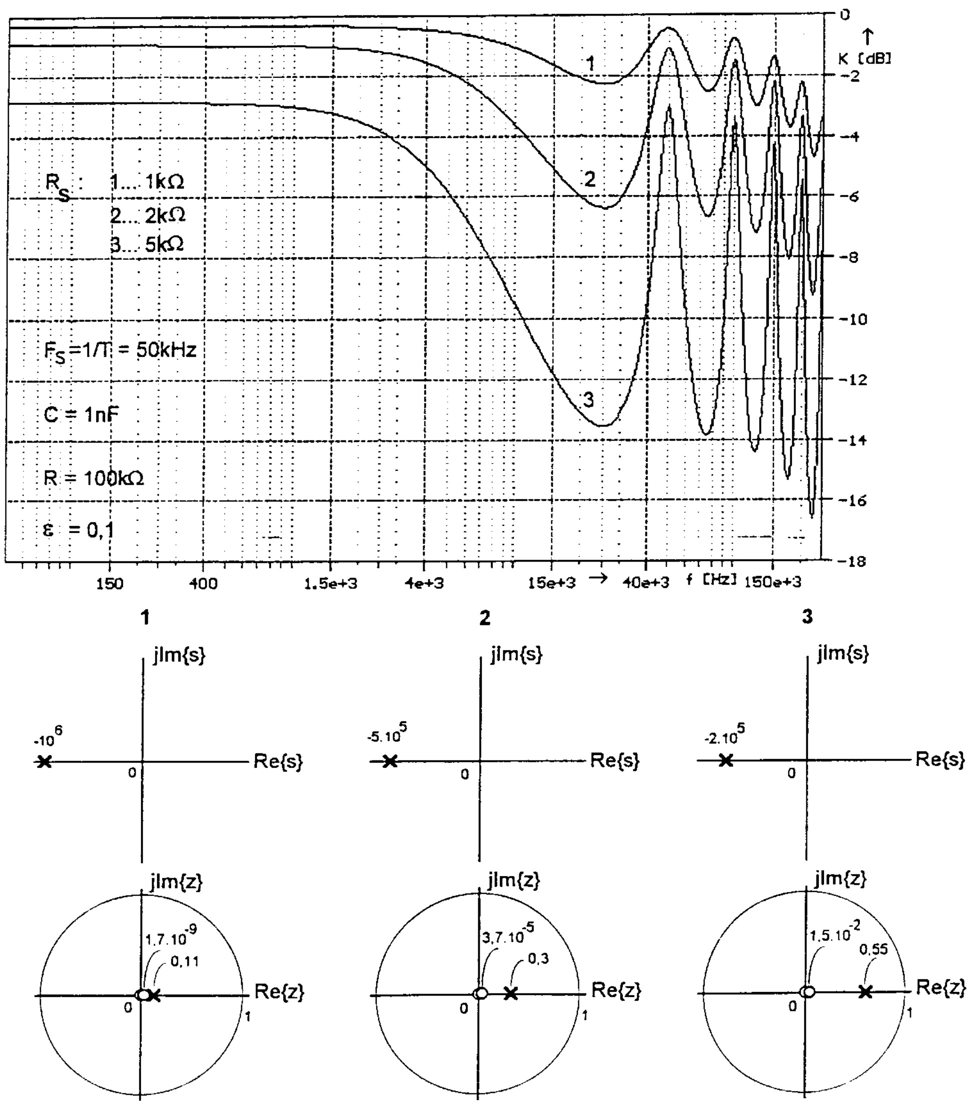


Fig. 3. Frequency responses of a sample-and-hold circuit, computed using the generalized transfer functions, with corresponding s and z pole and zero locations.

of the zero-pole locations in the s and z domains. Consider the SH circuit in Fig. 1. Equations (12) reduce to

$$\begin{aligned}(G + G_s + sC)V_1(s) &= Cv_2(kT^-) + G_s w(s)\\ (G + sC)V_2(s) &= Cv_1(kT + \varepsilon T)\end{aligned}$$

where $G = 1/R, G_s = 1/R_s$ is the switch-on conductance, and the switching phases are given the subscripts 1 (switch on) and 2 (switch off). The time-domain solution is

$$\begin{aligned}v_1(kT + \varepsilon T) &= g_1^* v_2(kT^-) + \int_0^{\varepsilon T} g(\xi) w(kT + \varepsilon T - \xi)\, d\xi\\ v_2(kT^-) &= v_1(kT - T + \varepsilon T^-) g_2^* \end{aligned} \tag{17}$$

where $g_1^* = e^{-\varepsilon T/\tau_1}$, $g_2^* = e^{-(T-\varepsilon T)/\tau_2}$, $g(t) = (1/\tau_1)e^{-t/\tau_1}$, $\tau_1 = R_s R/(R_s + R)C$ and $\tau_2 = RC$. Equations (17) yield the recurrent formula which can be solved using equivalent signals

$$v_\varepsilon^e(t) = g_1^* g_2^* v_\varepsilon^e(t - T) + \int_0^{\varepsilon T} g(\xi) w(t - k\xi) dk\xi.$$

After Laplace transformation and some rearrangements, the GTF (2) is obtained in the following closed form:

$$K_\varepsilon(s, z) = \frac{R}{R + R_s} \frac{1}{1 - z^{-1} g_1^* g_2^*} \frac{1 - g_1^* z^{-\varepsilon}}{1 + s\tau_1}. \tag{18}$$

This GTF models the sample selection dynamic error of the SH circuit. The transfer function is characterized by the s and z domain zero-pole locations. After substitution $s = j\omega$ and $z = \exp(j\omega T)$, the frequency responses in Fig. 3 are obtained. The zero-pole locations are also shown. The frequency responses are not periodic due to the s-domain poles. The periodic frequency response of the discrete-time system is caused by the aliasing effect. However, in case of a SH circuit, the internal circuit variables cannot follow the input signal for all frequencies due to the inertia of the input branch, and the resulting frequency responses converge to zero. The character of this convergence is given by the s-domain poles.

In time domain, the response can be observed by the convergence of the discrete-time signal defined in given equidistant points. The convergence character is defined by the z-domain poles of the GTF. Stability properties of the switched network can be investigated using locations of the z-domain poles.

Computer programs for switched network simulation based on the GTF idea have been developed [7], [8]. In the following, principles of computer simulation for two-phase switching are presented.

V. Circuit Equations and Preprocessing

Due to the universal character of GTF's, simulation covers all linear networks, switched or unswitched. This property is unusual, because most programs are designed to simulate only one type of switched circuits [9]–[13]. In the case of ideal switched-capacitor or switched-current networks, the resulting transfer function is only in the z-domain and the corresponding frequency response is periodical. For linear circuits without switches, s-domain transfer functions are obtained. General switched networks have transfer functions in both domains.

The two-graph modified nodal approach is used to set up the equations [13], [14]. We will consider only systems with two phases, with $T = T_1 + T_2$, leading to the sets

$$\begin{aligned}(\boldsymbol{G}_1 + s\boldsymbol{C}_1)\boldsymbol{V}_1 &= \boldsymbol{D}_1 W_1 + \boldsymbol{C}_{12}\boldsymbol{v}_2(kT^-)\\ (\boldsymbol{G}_2 + s\boldsymbol{C}_2)\boldsymbol{V}_2 &= \boldsymbol{D}_2 W_2 + \boldsymbol{C}_{21}\boldsymbol{v}_1(kT + T_1^-).\end{aligned} \tag{19}$$

Preprocessing of (19) will be done with the aim to derive closed-form expressions of generalized transfer functions. We concentrate on the numerical aspects. Formal solution of (19) is

$$\begin{aligned}\boldsymbol{V}_1(s) &= \boldsymbol{Y}_1^{-1}\boldsymbol{D}_1 W_1(s) + \boldsymbol{Y}_1^{-1}\boldsymbol{C}_{12}\boldsymbol{v}_2(kT^-)\\ \boldsymbol{V}_2(s) &= \boldsymbol{Y}_2^{-1}\boldsymbol{D}_2 W_2(s) + \boldsymbol{Y}_2^{-1}\boldsymbol{C}_{21}\boldsymbol{v}_1(kT + T_1^-)\end{aligned} \tag{20}$$

where

$$\boldsymbol{Y}_i = \boldsymbol{G}_i + s\boldsymbol{C}_i. \tag{21}$$

After the inverse Laplace transformation we obtain at the end of phase 1, at $t = kT + T_1^-$:

$$\begin{aligned}\boldsymbol{v}_1(kT + T_1^-) = &\int_0^{T_1} \boldsymbol{g}_1(\xi) w_1(kT + T_1 - \xi)\, d\xi\\ &+ \boldsymbol{g}_1^*(T_1)\boldsymbol{v}_2(kT^-)\end{aligned} \tag{22}$$

and at the end of phase 2 at $t = kT + T^-$:

$$\begin{aligned}\boldsymbol{v}_2(kT + T^-) = &\int_0^{T_2} \boldsymbol{g}_2(\xi) w_2(kT + T - \xi)\, d\xi\\ &+ \boldsymbol{g}_2^*(T_2)\boldsymbol{v}_1(kT + T_1^-)\end{aligned} \tag{23}$$

The matrices and vectors

$$\begin{aligned}\boldsymbol{g}_1(t) &= L^{-1}[\boldsymbol{Y}_1^{-1}\boldsymbol{D}_1] \quad \boldsymbol{g}_1^*(t) = L^{-1}[\boldsymbol{Y}_1^{-1}\boldsymbol{C}_{12}]\\ \boldsymbol{g}_2 &= L^{-1}[\boldsymbol{Y}_2^{-1}\boldsymbol{D}_2] \quad \boldsymbol{g}_2^*(t) = L^{-1}[\boldsymbol{Y}_2^{-1}\boldsymbol{C}_{21}]\end{aligned} \tag{24}$$

can be obtained using the algorithm of inverse Laplace transform [14]–[16].

Rearranging (22) and (23) yields formulas for the vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$:

$$\begin{aligned}&\boldsymbol{v}_1(kT + T_1^-)\\ &\quad= \int_0^{T_1} \boldsymbol{g}_1(\xi) w_1(kT + T_1 - \xi)\, d\xi\\ &\qquad+ \boldsymbol{g}_1^*(T_1)\int_0^{T_2} \boldsymbol{g}_2(\xi) w_2(kT - \xi)\, d\xi\\ &\qquad+ \boldsymbol{g}_1^*(T_1)\boldsymbol{g}_2^*(T_2)\boldsymbol{v}_1(kT + T_1^- - T)\end{aligned} \tag{25}$$

$$\begin{aligned}&\boldsymbol{v}_2(kT^-)\\ &\quad= \int_0^{T_2} \boldsymbol{g}_2(\xi) w_2(kT - \xi)\, d\xi\\ &\qquad+ \boldsymbol{g}_2^*(T_2)\int_0^{T_1} \boldsymbol{g}_1(\xi) w_1(kT - T_1 - T - \xi)\, d\xi\\ &\qquad+ \boldsymbol{g}_2^*(T_2)\boldsymbol{g}_1^*(T_1)\boldsymbol{v}_2(kT + T^-).\end{aligned} \tag{26}$$

Equations (25) and (26) are simplified versions of the general equation (6). Discrete-time signals $\boldsymbol{v}_1(kT + T_1^-)$ and $\boldsymbol{v}_2(kT^-)$ are replaced by equivalent continuous time signals $\boldsymbol{v}_1^e(t)$ and $\boldsymbol{v}_2^e(t)$

$$\begin{aligned}\boldsymbol{v}_1^e(t) &= \boldsymbol{v}_1(kT + T_1^-)|_{kT+T_1=t} \to \boldsymbol{V}_1^e(s)\\ \boldsymbol{v}_2^e(t) &= \boldsymbol{v}_2(kT^-)|_{kT=t} \to \boldsymbol{V}_2^e(s)\end{aligned}$$

Symbols $\boldsymbol{V}_1^e(s)$ and $\boldsymbol{V}_2^e(s)$ denote Laplace transforms of these equivalent signals. They can be constructed for the input signals in a similar way

$$\begin{aligned}w_1^e(t) &= w_1(kT + T_1^-)|_{kT+T_1=t} \to W_1^e(s)\\ w_2^e(t) &= w_2(kT^-)|_{kT=t} \to W_2^e(s).\end{aligned}$$

General solution has the form

$$\begin{aligned}\boldsymbol{V}_1^e &= \boldsymbol{K}_{11}W_1^e(s) + \boldsymbol{K}_{12}W_2^e(s)\\ \boldsymbol{V}_2^e &= \boldsymbol{K}_{21}W_1^e(s) + \boldsymbol{K}_{22}W_2^e(s)\end{aligned} \tag{27}$$

which is (15) for a two-phase switched circuit. The generalized transfer functions $\boldsymbol{K}_{ij}$ depend on the input signal. We can distinguish the following cases:

1) The signal w_1 (w_2) has a spectrum that after sampling with the sampling frequency $f_s = 1/T = 1/(T_1 + T_2)$, the overlapping of the nonzero original and of the periodized spectral components will not occur. The simplest case is for $f_{\max} < f_s/2$. These signals and the equivalent ones are then identical, $w_1 = w_1^e, w_2 = w_2^e$, and

$$\begin{aligned}\boldsymbol{K}_{11}(s,z) &= [\boldsymbol{E} - \boldsymbol{F}_1 z^{-1}]^{-1}\boldsymbol{G}_1(s,z)\\ \boldsymbol{K}_{12}(s,z) &= [\boldsymbol{E} - \boldsymbol{F}_1 z^{-1}]^{-1}\boldsymbol{g}_1^*(T_1)\boldsymbol{G}_2(s,z)z^{-T_1/T}\\ \boldsymbol{K}_{21}(s,z) &= [\boldsymbol{E} - \boldsymbol{F}_2 z^{-1}]^{-1}\boldsymbol{g}_2^*(T_2)\boldsymbol{G}_1(s,z)z^{-T_2/T}\\ \boldsymbol{K}_{22}(s,z) &= [\boldsymbol{E} - \boldsymbol{F}_2 z^{-1}]^{-1}\boldsymbol{G}_2(s,z)\end{aligned} \tag{28}$$

where

$$\boldsymbol{F}_1 = \boldsymbol{g}_1^*(T_1)\boldsymbol{g}_2^*(T_2), \quad \boldsymbol{F}_2 = \boldsymbol{g}_2^*(T_2)\boldsymbol{g}_1^*(T_1) \tag{29}$$

$$\boldsymbol{G}_i(s,z) = \int_0^{T_i} \boldsymbol{g}_i(\xi)e^{-s\xi}\, d\xi \qquad i = 1,2 \tag{30}$$

$$z = e^{sT}. \tag{31}$$

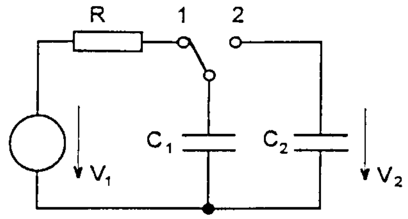


Fig. 4. Example of a switched network $R = 1\ \mathrm{M}\Omega, C_1 = 1$ nF, $C_2 = 10$ nF, $T = 10\ \mu$s, $T_1 = T_2 = T/2, R_1 = R_2 = 500\ \Omega$ (switch-on resistances).

In the above $\boldsymbol{E}$ denotes the unit matrix. After substitution $s = j\omega$ and $z = \exp(j\omega T)$, frequency responses which are not periodical can be obtained, due to the integrals $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$.

2) The switched circuit is excited by a single signal $w_1 = w_2 = w$. The spectrum of this signal must satisfy conditions from point 1). The solution is the same one as in 1) but (27) are now in the form

$$\begin{aligned}\boldsymbol{V}_1^e &= (\boldsymbol{K}_{11} + \boldsymbol{K}_{12})W^e(s)\\ \boldsymbol{V}_2^e &= (\boldsymbol{K}_{21} + \boldsymbol{K}_{22})W^e(s)\end{aligned} \tag{32}$$

3) The signals w_1 and w_2 are sampled and held,

$$\begin{aligned}w_1(t) &= w_1^e(kT) \quad \text{for} \quad t \in \langle kT, kT + T_1)\\ w_2(t) &= w_2^e(kT + T_1) \quad \text{for} \quad t \in \langle kT + T_1, kT + T)\end{aligned}$$

In this case (27) remains valid. The following equations for the GTF now apply

$$\begin{aligned}\boldsymbol{K}_{11}(z) &= [\boldsymbol{E} - \boldsymbol{F}_1 z^{-1}]^{-1}\boldsymbol{G}_1 z^{-T_1/T}\\ \boldsymbol{K}_{12}(z) &= [\boldsymbol{E} - \boldsymbol{F}_1 z^{-1}]^{-1}\boldsymbol{g}_1^*(T_1)\boldsymbol{G}_2 z^{-1}\\ \boldsymbol{K}_{21}(z) &= [\boldsymbol{E} - \boldsymbol{F}_2 z^{-1}]^{-1}\boldsymbol{g}_2^*(T_2)\boldsymbol{G}_1 z^{-1}\\ \boldsymbol{K}_{22}(z) &= [\boldsymbol{E} - \boldsymbol{F}_2 z^{-1}]^{-1}\boldsymbol{G}_2 z^{-T_2/T}\end{aligned} \tag{33}$$

where

$$\boldsymbol{G}_i = \int_0^{T_i} \boldsymbol{g}_i(\xi)\, d\xi \qquad i = 1, 2 \tag{34}$$

are the vectors of step responses in phases 1 and 2, at time points T_1 and T_2. When the input signal is sampled and held, the transmission properties of the switched circuit are given only by the functions of the z operator and frequency responses will be periodical.

4) Single staircase input signal: see point 3) where $w_1^e = w_2^e$. The solution is the same as in 3) with $w_1^e = w_2^e$.

5) Single sample and held input signal with the hold regime over both switching phases,

$$w(t) = w^e(kT) \quad \text{for} \quad t \in \langle kT, kT + T).$$

In this case

$$\begin{aligned}\boldsymbol{V}_1^e &= [\boldsymbol{E} - \boldsymbol{F}_1 z^{-1}]^{-1}[\boldsymbol{G}_1 + \boldsymbol{g}_1^*(T_1)\boldsymbol{G}_2 z^{-1}]z^{-T_1/T}\boldsymbol{V}^e\\ \boldsymbol{V}_2^e &= [\boldsymbol{E} - \boldsymbol{F}_2 z^{-1}]^{-1}[\boldsymbol{G}_2 + \boldsymbol{g}_2^*(T_2)\boldsymbol{G}_1]z^{-1}\boldsymbol{V}^e\end{aligned} \tag{35}$$

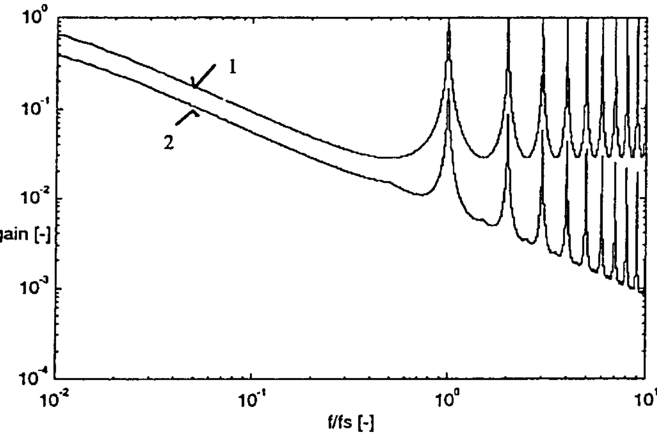
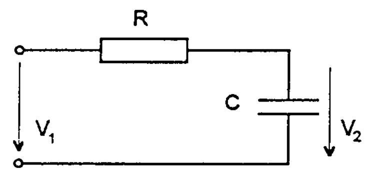
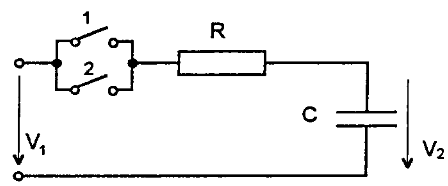


Fig. 5. Frequency responses of network in Fig. 4: curve 1: ideal, all resistances are zero, curve 2: with resistances.



(a)



(b)

Fig. 6. An unswitched network, simulated as a switched network. $R = 1\,\mathrm{k}\Omega, C = 1\,\mu\mathrm{F}, T = 1\,\mu\mathrm{s}, T_1 = T_2 = T/2$.

VI. Semisymbolic Analysis of GTF

The form of the GTF depends on the character of input signal, see (28), (32), (33), and (35). Semisymbolic analysis can be composed of semisymbolic inversion of the matrices $\boldsymbol{E} - \boldsymbol{F}_i z^{-1}$ and semisymbolic expression of the vectors $\boldsymbol{G}_i$ in terms of operators z and s.

1) Semisymbolic inversion of matrices $\boldsymbol{E} - \boldsymbol{F}_i z^{-1}$. It can be done so that

$$[\boldsymbol{E} - \boldsymbol{F} z^{-1}]^{-1} = \frac{\boldsymbol{A}_0 + \boldsymbol{A}_1 z^{-1} + \cdots + \boldsymbol{A}_n z^{-n}}{a_0 + a_1 z^{-1} + \cdots + a_n z^{-n}} \tag{36}$$

where n is the dimension of the matrix $\boldsymbol{F}$, the denominator is its determinant, and $\boldsymbol{A}_k$ are matrices. The expansion can be done by means of the QZ or QR algorithms [14], [17].

2) Semisymbolic analysis of $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ in mixed s and z domains. For dc transmission we insert $s = 0$ and (30) reduces to (34). This is a vector of the step response at the time instance T_i for phase i. For nonzero frequency we obtain a function of both operators. The vector $\boldsymbol{g}_i$ represents the impulse response in phase i. During this phase, the switched network is a stationary system described by (20). The impulse response is defined as

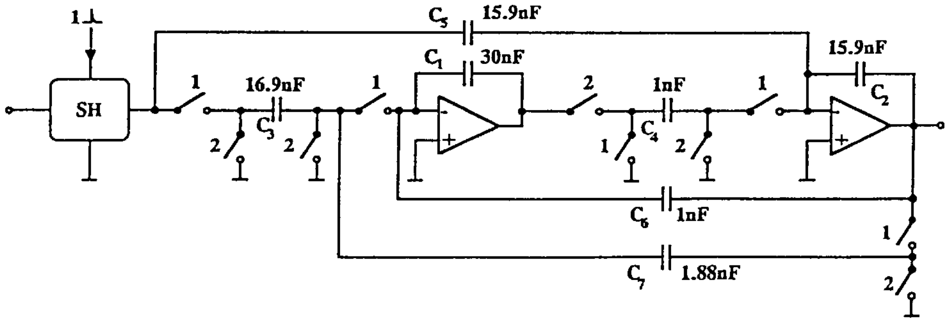


Fig. 7. Example of an SC filter.

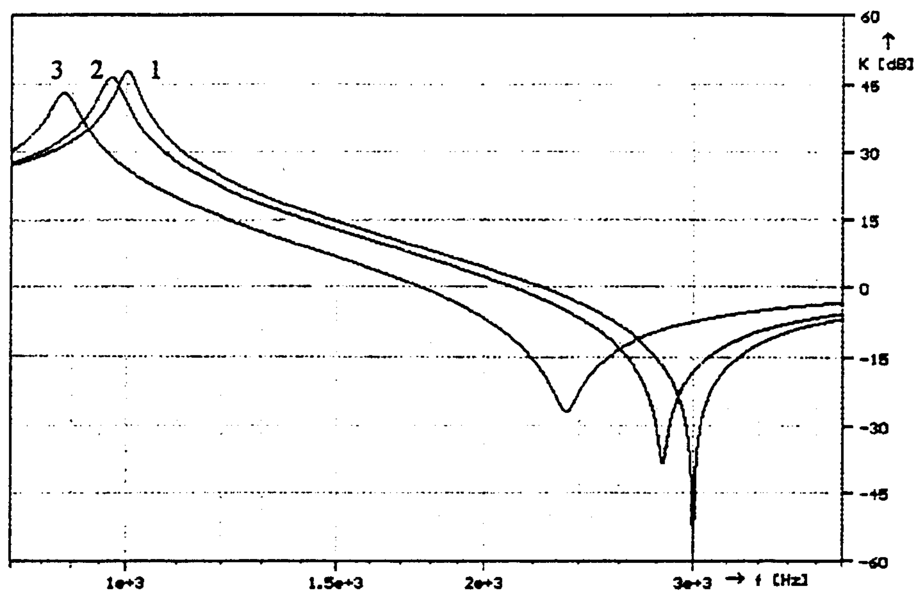


Fig. 8. Frequency responses of the network in Fig. 7. The sampling frequency is 100 kHz. Switch resistances are 0 for curve 1, 50 Ω for curve 2, 100 Ω for curve 3.

zero-state response. If the exciting signal is a Dirac impulse, we have

$$\boldsymbol{Y}_i\boldsymbol{V}_i = \boldsymbol{D}_i \qquad i = 1, 2 \tag{37}$$

where $\boldsymbol{V}_i$ is now a vector of the impulse response in operator form. In other words, $\boldsymbol{V}_i$ is the vector of the transfer function of switched network in phase i

$$\boldsymbol{V}_i = \boldsymbol{K}_i \qquad i = 1, 2. \tag{38}$$

Each element of the vector $\boldsymbol{K}_i$ is a rational function of the operator s. Coefficients of this function can be found by applying the QR algorithm to the matrix equation (20) for $W_1 = W_2 = 0$.

We will derive the integral (30) using the transfer function. The impulse response $\boldsymbol{g}_i$ can be calculated as the inverse Laplace transformation

$$\boldsymbol{g}_i(\xi) = \frac{1}{2\pi j}\int_c \boldsymbol{K}_i(p)e^{p\xi}\,dp \qquad i = 1, 2 \tag{39}$$

where the symbol $\int_c$ denotes a contour integral. Substituting (39) into (30), we obtain after some rearrangements

$$\begin{aligned}\boldsymbol{G}_i(s, z) &= \frac{1}{2\pi j}\int_0^{T_i}\int_c \boldsymbol{K}_i(p)e^{p\xi}e^{-s\xi}\,dp\,d\xi\\ &= \frac{1}{2\pi j}\int_c \boldsymbol{K}_i(p)\frac{e^{(p-s)T_i}-1}{p-s}\,dp \qquad i = 1, 2\end{aligned}$$

Consider the mth element of $\boldsymbol{K}_i(p)$. It is a rational function

$$\frac{A_i(p)}{B_i(p)} = \frac{a_0 + a_1 p + \cdots + a_{n_i}p^{n_i}}{b_0 + b_1 p + \cdots + b_{n_i}p^{n_i}}. \tag{40}$$

The denominator is the same for all elements of the vector $\boldsymbol{K}_i(p)$ and its roots are the poles of the switched system in phase i. The corresponding mth element of $\boldsymbol{G}_i$ is

$$G_i = \frac{1}{2\pi j}\int_c \frac{A_i(p)}{B_i(p)}\frac{e^{(p-s)T_i}-1}{p-s}\,dp \qquad i = 1, 2 \tag{41}$$

The integral can be solved using residue calculus. To simplify the steps, consider simple poles of the system during phase i. Equation (40) can be decomposed into partial fractions

$$\frac{A_i(p)}{B_i(p)} = \sum_{k=1}^{n^i}\frac{R_k}{p-p_k} \tag{42}$$

where p_k is a pole and R_k its residue. Selecting proper contour of integration we get

$$\begin{aligned} G_i &= \frac{1}{2\pi j}\int_c \sum_{k=1}^{n_i} \frac{R_k}{p-p_k}\frac{e^{(p-s)T_i}-1}{p-s}\,dp \\ &= \sum_{k=1}^{n_i} R_k \frac{1-e^{(p_k-s)T_i}}{s-p_k} \\ &= \frac{A_i(s)}{B_i(s)} - e^{-sT_i}\sum_{k=1}^{n_i}\frac{R_k}{s-p_k}e^{p_kT_i} \\ &= \frac{A_i(s)-z^{-T_i/T}C_i(s)}{B_i(s)} \qquad i=1,2. \end{aligned} \tag{43}$$

The poles p_k and coefficients of the polynomials A_i and B_i can be found using the QR algorithm. Coefficients of the polynomial C_i can be obtained using the poles and R_k. Using these steps, we calculate G_i in a semisymbolic form.

VII. EXAMPLES

We will demonstrate the program on examples. All semisymbolic and numerical results are outputs of the program SPIN [8].

Example 1 — Semisymbolic Analysis of Ideal and Nonideal Switched Capacitor Circuits: Consider the SC lowpass filter in Fig. 4. Nonzero switch-on resistances R_1 and R_2 in phases 1 and 2 are taken into account. Our aim is to obtain the input–output transfer function by considering input and output samples in phase 1. In an ideal case, $R = R_1 = R_2 = 0$. The transfer function is completely described in the z-domain

$$K(z) = \frac{C_1}{C_1+C_2}\frac{z^{-1}}{1-\dfrac{C_2}{C_1+C_2}z^{-1}}$$

and our system is modeled as a first-order discrete-time system with a periodical frequency response shown in Fig. 5, curve 1. For this ideal case, our algorithm gives

$$K(s,z) = \frac{0.090909090909z^{-1}}{1-0.909090909091z^{-1}} \times 1 = K(z)$$

where multiplication by 1 denotes the value of $G_1(s,z)$. This can be verified by inserting numerical values.

Analysis of GTF's for a nonideal switched network with all resistances leads to the mixed s-z description shown at the bottom of the page.

The second fraction denotes $G_1(s,z)$. Resistors increase the z-domain order to 2 and add one s-domain pole. The resulting frequency response is not periodic any more—see Fig. 5, curve 2.

Example 2 — Analysis of Both Switched- and Continuous-Time Circuits: For a continuous-time network, the resulting transfer function will have only the operator s. According to the superposition theorem, excitation of an unswitched network can be considered as the superposition of nonoverlapping fragments of input signals in phases 1 and 2. Transmission of these input fragments to the output is described mathematically by GTF's K_{11}, K_{12}, K_{21}, and K_{22}, see (27). Interpretation of the equivalent signal [5], [6] leads to the conclusion that classical transfer function of an unswitched network can be calculated from partial GTF's as follows:

$$K(s) = K_{11}(s,z)+K_{12}(s,z) = K_{21}(s,z)+K_{22}(s,z). \tag{44}$$

We will analyze the RC network in Fig. 6(a). Its s-domain transfer function is

$$K(s) = \frac{1}{1+sRC}$$

The algorithm based on the GTF needs the switching period T, but the result of (44) will not depend on its value. Switches in Fig. 6(b) operate so that they eliminate each others influence and model fragmentation of the input signal. The algorithm provides the following solution:

$$\begin{aligned} K_{11}(s,z) = K_{22}(s,z) &= \frac{1}{1-0.999\,000\,499\,833z^{-1}} \\ &\quad\cdot \frac{1-0.999\,500\,124\,979z^{-1/2}}{1+10^{-3}s} \\ K_{12}(s,z) = K_{21}(s,z) &= \frac{0.999\,500\,124\,979z^{-1/2}}{1-0.999\,000\,499\,833z^{-1}} \\ &\quad\cdot \frac{1-0.999\,500\,124\,979z^{-1/2}}{1+10^{-3}s} \\ K_{11}(s,z)+K_{12}(s,z) & \\ = K_{21}(s,z)+K_{22}(s,z) &= \frac{1}{1+10^{-3}s} = K(s) \end{aligned}.$$

Example 3 — Verification of Semisymbolic Analysis, Influence of Switch Resistances: The SC lowpass elliptic filter in Fig. 7 has frequency rejection at $0.03f_s$ and has the resonance maximum at $0.01f_s$. The sampling frequency is $f_s = 100$ kHz. The filter was designed for ideal elements, but laboratory measurements have shown considerable differences from the required response, due to actual properties of switches. Computation of the z-domain system functions was performed in [2]. All system functions have the same denominator, independent of the choice of input and output samples. Its symbolic form is

$$\det = 1 + a_1z^{-1} + a_2z^{-2}$$

with

$$a_1 = \frac{C_4(C_6+C_7)}{C_1C_2} - 2 \quad a_2 = 1 - \frac{C_4C_6}{C_1C_2}.$$

$$K(s,z) = \frac{0.090907572573z^{-1}}{1-0.945724937524z^{-1}+6.728953601465\times10^{-6}z^{-2}}\frac{1-0.402890321529z^{-1/2}}{1+5.5\times10^{-6}s}$$

Comparison of the semisymbolic result agreed with exact values. The analyzes have shown significant influence of the switch resistances. Corresponding frequency responses are in Fig. 8. For semiconductor switches with typical resistances of about 100 Ω, the real response will differ significantly.

VIII. Conclusion

A novel algorithm for mixed s-z domain semisymbolic analysis of general linear networks has been presented. The algorithm is based on the theory of generalized transfer functions and includes both classical s-domain transfer functions of continuous-time networks and z-domain transfer functions of discrete-time networks. In case of general switched networks, the resulting transfer function has both operators, to model simultaneously the mixed continuous- and discrete-time character. The algorithm can be used for efficient semisymbolic analysis of classical and mixed-mode systems.

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