Modeling of switched DC-DC converters by mixed s-z description

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***Abstract*— The paper explains the basic ideas of how to model the dynamical properties of switched DC-DC converters by means of the so-called generalized transfer functions (GTFs). Switched DC-DC converters like bust or bust-buck types exhibit some properties of continuous- and discrete-time characters. These systems can be globally modeled by mixed s-z description, utilizing both the Laplace and *z*-transform operators. Then the resulting line-to-output and control-to-output transfer functions and the corresponding frequency responses can model some system behavior more faithfully than the classical s-domain zeros/poles location, including special effects above Nyquist's frequency.**

I. INTRODUCTION

Frequency-domain modeling of dynamical properties of switched DC-DC converters is based on the so-called averaging approach [1]. Several methods are currently employed, depending on which of the converter subcircuits should be included in the averaging process: if only the PWM switch (see the Vorpérian model of PWM switch) [2], or the PWM switch together with the inductor (the so-called Switched Inductor Model, SIM) [3], etc. Different approaches should be applied to the modeling of propagation of the input disturbances to the output (the so-called line-to-output transfer function), and propagation of duty ratio perturbation to the output (the so-called control-to-output transfer function). The modeling also depends on whether the converter operates in the continuous current mode (CCM) or the discontinuous current mode (DCM), and whether the classical voltage mode control or the current mode control is applied. In particular, the modeling of the last mentioned regulators is rather complicated. For a more complex description of the dynamic features and for monitoring the stability, special approximate methods were developed [4], which model the influence of switching processes on the loop gain, particularly in the frequency region in the vicinity of Nyquist's frequency $f_{\text{switch}}/2$.

The switched DC-DC converter can be considered an analog linear time-varying system, where the externally controlled switches are modeled by time-varying resistances. To model such systems, the so-called generalized transfer functions (GTFs) [5], [6] can be applied. GTFs are functions of two well-known *s* and *z* operators, simultaneously describing both the continuous-time and discrete-time behavior of switched converter. Frequency responses are obtained after double substitution $s=j\omega$ and $z=\exp(j\omega T)$, where T is the switching period. The GTF-based modeling offers a more precise description of converter dynamics than the presently used approximate procedures, based only on the *s*- or *z*-domain transfer functions.

II. GTF OF SWITCHED DC-DC CONVERTERS

Some basic types of the well-known switched DC-DC converters are summarized in Figs. 1(a-c). All of them are two-phase switched circuits with a switching diagram as in Fig. (d), when the active switch S causes by its ON/OFF operation inverse states of the passive switch D, i.e. OFF/ON. The relative duration of the ON state of active switch with respect to the switching period is called the duty ratio (D). This quantity is usually controlled depending on the output voltage, which is then stabilized via pulse width modulation. After neglecting the nonlinear operation of switching devices, the converter can be described in each of its switching phases by a linear model with dominant couple of state variables – voltage across the capacitor and the inductor current.

Let us consider the following assumptions: Linear model in each switching phase, continuous conducting mode, variable duty ratio $D_k = D_0 + d_k$, $d_k << D_0$. Input voltage V will have a DC component V_0 and a superposed AC component $v(t) << V_0$. The aim of the analysis is to get the frequency dependence of the AC component of the output voltage and the other observed signals without limitation of the bandwidth of signal $v(t)$.

Due to the switching processes, the circuit variables of switched converters are not smooth time domain functions. According to the theory of generalized transfer functions [5],

This work has been supported by the Grant Agency of the Czech Republic under grant No. 102/05/P001, and by the research programmes of Brno University of Technology MSM0021630503 and MSM0021630513. Special thanks are due for the support by National Semiconductor, Santa Clara, USA.

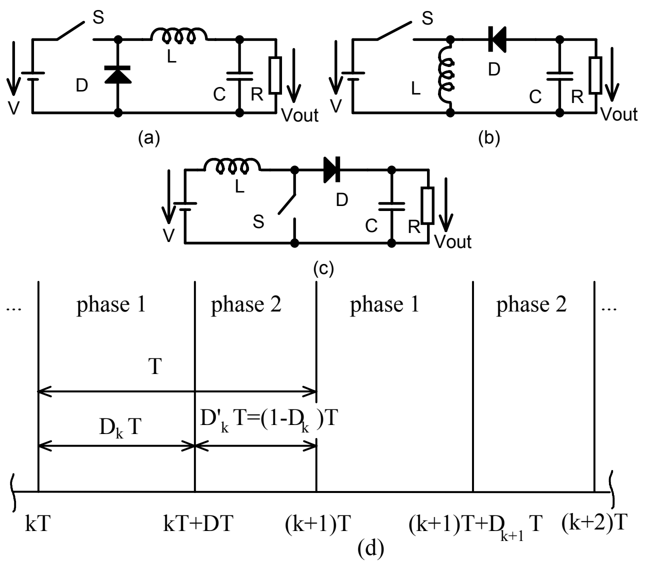

Figure 1. Fundamental circuit diagrams of DC-DC converters, types: (a) buck, (b) boost, (c) buck-boost; the switching diagram (d).

these signals can be represented by smooth, so-called equivalent signals which fulfill the two following conditions: (1) the original and equivalent signals have identical values at the sampling instances that we are interested in (mostly on the boundaries of switching phases), (2) the bandwidth of the equivalent signals coincides with the bandwidth of input signal. The first condition stands that the equivalent signals form abstract envelopes of real signals. In case of negligible influence of switching processes, there is a good compliance between the real signals and their envelopes. The second condition enables comparing equivalent signals and input signal in the frequency domain by means of linear theory. In other words, it enables us to define transfer functions.

Let us observe the converter state by vector $\mathbf{X}$, which consists – in the simplest case – of a couple of state variables V_C and I_L. All remaining circuit currents or voltages can be obtained by a linear combination of these state variables and input voltage. In the following, we will assume a continuity of state variables both in the frame of switching phases and at transition instances between them.

Considering the above assumptions, the switched converter can be described by the following equations:

End of switching phase 1 – $t = kT+D_kT$:

$$\mathbf{X}(kT+D_kT) = \mathbf{G}_1(D_kT)\mathbf{X}(kT) + \int_0^{D_kT} \mathbf{g}_1(\xi)V(kT+D_kT-\xi)d\xi \cdot \quad (1)$$

End of switching phase 2 – $t = kT+T$:

$$\mathbf{X}(kT+T) = \mathbf{G}_2(D_k'T)\mathbf{X}(kT+D_kT) + \int_0^{D_k'T} \mathbf{g}_2(\xi)V(kT+T-\xi)d\xi \cdot \quad (2)$$

The variables used in (1) and (2) are as follows:

$\mathbf{G}_1$, $\mathbf{G}_2$… matrices of converter natural responses within phase 1 and phase 2, respectively, defined on the assumption of zero-input signal V,

$\mathbf{g}_1$, $\mathbf{g}_2$… matrices of converter impulse responses within phase 1 and phase 2, respectively, defined on the assumption of input signal V as a Dirac impulse, operating at the beginning of switching phase.

Now regard the input voltage and the duty ratio as a sum of DC and AC components. On the assumption of $v<<V_0$ and $d_k<<D_0$, due to the circuit linearity, the state vector $\mathbf{X}$ will be made up of a constant and a variable term, i.e. $\mathbf{X}(kT) = \mathbf{X}_0 + \mathbf{x}(kT)$. This result can be obtained after perturbing the individual quantities in equations (1) and (2) and neglecting insignificant terms. The final equations are in the form

$$\mathbf{X}_0 = [\mathbf{I}-\mathbf{G}]^{-1}[\mathbf{G}_2\mathbf{H}_1 + \mathbf{H}_2]V_0, \quad (3)$$

$$\mathbf{x}(kT+T) = \mathbf{G}\mathbf{x}(kT) + \mathbf{F}d_kT + \\ + \mathbf{G}_2\int_0^{D_0T}\mathbf{g}_1(\xi)v(kT+D_0T-\xi)d\xi + \int_0^{D_0'T}\mathbf{g}_2(\xi)v(kT+T-\xi)d\xi. \quad (4)$$

For simplicity, the following notation is used in (3) and (4):

$$\mathbf{G} = \mathbf{G}_2\mathbf{G}_1, \quad (5)$$

where $\mathbf{G}_2 = \mathbf{G}_2(D_0'T)$, $\mathbf{G}_1 = \mathbf{G}_1(D_0T)$. Equation (3) describes computing the steady-state vector $\mathbf{X}$ at ends of switching phases 2, when the DC-DC converter is excited only by DC voltage V_0. Equation (4) is a special "integral-difference" formula describing the converter dynamics. Matrix (5) predicts the system stability.

$\mathbf{I}$ is the unity matrix. $\mathbf{H}_1$ and $\mathbf{H}_2$ are vectors of converter forced responses to constant input signal V_0 within phase 1 and phase 2, respectively, considering zero initial state:

$$\mathbf{H}_1 = \int_0^{D_0T}\mathbf{g}_1(\xi)d\xi \,, \mathbf{H}_2 = \int_0^{D_0'T}\mathbf{g}_2(\xi)d\xi \cdot \quad (6)$$

Vector $\mathbf{F}$ is given by the following equation:

$$\mathbf{F} = (\mathbf{G}_2\mathbf{G}_{d1} - \mathbf{G}_{d2}\mathbf{G}_1)\mathbf{X}_0 + (\mathbf{G}_2\mathbf{H}_{d1} - \mathbf{G}_{d2}\mathbf{H}_1 - \mathbf{H}_{d2})V_0, \quad (7)$$

where subscripts d denote the derivatives of the corresponding matrices or vectors according to the variable D_0T.

Eq. (4) can be translated into the operator form by means of the theory of equivalent signals and GTFs. There are two sources of signal excitation: input signal $v(t)$ and perturbation d_k of duty ratio, which is induced by $v(t)$ through the feedback control. Let us consider harmonic excitation in general operator form (8). In the steady state, state vector $\mathbf{x}(kT)$ will be also represented by samples of harmonic signal:

$$v(t) = \hat{V}e^{st}, \; d_k = \hat{D}e^{skT}, \; \mathbf{x}(kT) = \hat{\mathbf{X}}e^{skT}. \quad (8)$$

The symbols $\hat{V}, \hat{D}, \hat{\mathbf{X}}$ represent the complex amplitudes of equivalent signals, which correspond to signals v, d, and $\mathbf{x}$. Note that continuous-time harmonic signal v is identical to its equivalent signal [5].

Substituting (8) into (4) and a rearrangement will yield

$$\hat{\mathbf{X}} = \mathbf{K}_d(z)\hat{D} + \mathbf{K}_v(s,z)\hat{V}\ , \tag{9}$$

where $\mathbf{K}_d$ and $\mathbf{K}_v$ are, respectively, the control-to-output and the line-to-output GTFs of the DC-DC converter according to Eqs (10) and (11):

$$\mathbf{K}_d(z) = (\mathbf{I} - \mathbf{G}z^{-1})^{-1}\mathbf{F}Tz^{-1} \tag{10}$$

$$\mathbf{K}_v(s,z) = (\mathbf{I} - \mathbf{G}z^{-1})^{-1}[\mathbf{G}_2\hat{\mathbf{H}}_1(s)z^{-D_0'} + \hat{\mathbf{H}}_2(s)] \tag{11}$$

Together with the operator of z-transform $z = e^{sT}$, the quantities

$$\hat{\mathbf{H}}_1(s) = \int_0^{D_0T}\mathbf{g}_1(t)e^{-st}dt\ ,\ \hat{\mathbf{H}}_2(s) = \int_0^{D_0'T}\mathbf{g}_2(t)e^{-st}dt \tag{12}$$

figure in the equations as the running Laplace transforms of converter impulse responses within phases 1 and 2, respectively. These functions generate the s-domain poles of converter frequency responses.

Equations (10) and (11) describe the generalized transfer functions of switched converter. Applying the double substitution $s = \mathrm{j}\omega$, $z = e^{\mathrm{j}\omega T}$ yields control-to-output and line-to-output frequency responses for an arbitrary bandwidth, i.e. without limitation to one half of the switching frequency, as is usual for most conventional procedures based on the averaging approach.

III. POSSIBILITIES OF COMPUTER SIMULATION

The above modeling can be utilized for computer simulation of converter, either by programming the equations into MATLAB-type software packages or by direct implementation into the SPICE-family simulation programs. In the first case, and also partly in the second one, it is advantageous to do analytical equation preprocessing for concrete converter topology.

Matrices $\mathbf{G}_1$, $\mathbf{G}_2$ and vectors $\mathbf{H}_1$ and $\mathbf{H}_2$ can be determined by numerical analyses of natural and forced responses of the converter separately during each quasistationary switching phase. For example, we can run multiple simulation from single PSPICE circuit file or process sequential runs with data sharing. The second procedure has been used for simulation by the CIA program [7]. The $\mathbf{G}$ matrix, whose eigenvalues are the z-domain poles of converter, can be simply computed from (5) or by transient analysis across two consecutive switching phases. DC steady state is preferably accessible directly by formula (3) instead of classical successive transient analysis.

Eq. (4) provides guidelines for numerical computation of vector $\mathbf{F}$: In the first step, transient analysis across both switching phases is performed from initial state $\mathbf{X}_0$. Constant input voltage V_0 is considered during the analysis, as well as the duty ratio, which is deflected from bias value D_0 by a difference $d \ll D_0$. As a result, a vector $\mathbf{X}$ is obtained. After its subtraction from vector $\mathbf{X}_0$ we have directly the product of vector $\mathbf{F}$ and the term dT.

Numerical computation of vectors (12) is more complicated. The details are described in [5]. Considering two dominant state variables in the converter, it is possible to have a go at analytical preprocessing of results and their subsequent programming. Note that in some converter topologies, when the state variables are not directly affected by input voltage within some switching phase, the corresponding vectors $\mathbf{H}$ and $\hat{\mathbf{H}}$ are simply zero. For instance, vectors $\mathbf{H}_2$ and $\hat{\mathbf{H}}_2$ will be zero for the buck and boost converter topologies.

PSPICE simulations indicate the crucial role of numerical precision in transient analysis for a correct determination of matrix characteristics of converter. Since the PSPICE program has no algorithm for finding directly the steady state and does not support multiple analyses with mutual data sharing, the simulation has been performed using the CIA program. To support also simulation in PSPICE, a subcircuit containing the behavioral model of GTFs (10), (11) has been developed by virtue of symbolic equations. These equations were obtained by the SNAP program [8] for symbolic analysis. Finally, these models were implemented as MATLAB m-files. As a result, these three different tools for modeling buck converter offer equivalent simulation results.

IV. DEMONSTRATION OF COMPUTER SIMULATION OF BUCK CONVERTER

Examples of PSPICE simulation of buck converter in Fig. 2 are given in [1]. An input voltage of 28V is converted into an output voltage of 15.2V, and the corresponding duty ratio is $D_0 = 0.543$. The switching frequency is 100 kHz. For a load resistance of 3Ω, the converter operates in the CCM. The modeling in [1] does not include ESRs of the inductor and the capacitor. The switches were considered to be ideal.

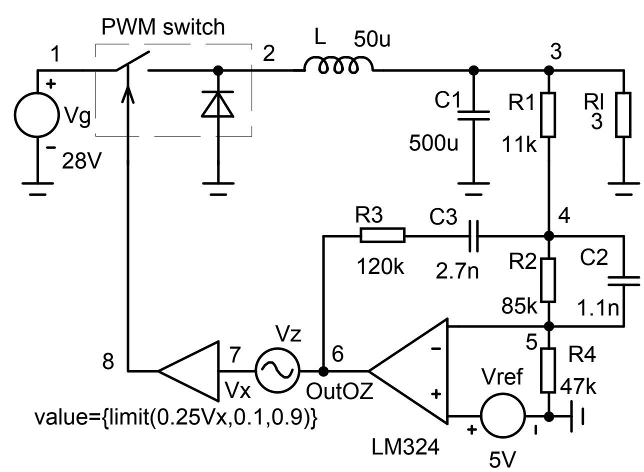

Figure 2. Model of a buck regulator according to [1].

Fig. 3 shows the results of PSPICE simulations of two models: the model based on the "averaging" technique [9], and the model based on the hybrid s-z modeling (GTF). In addition, these models include 50mΩ as capacitor ESR.

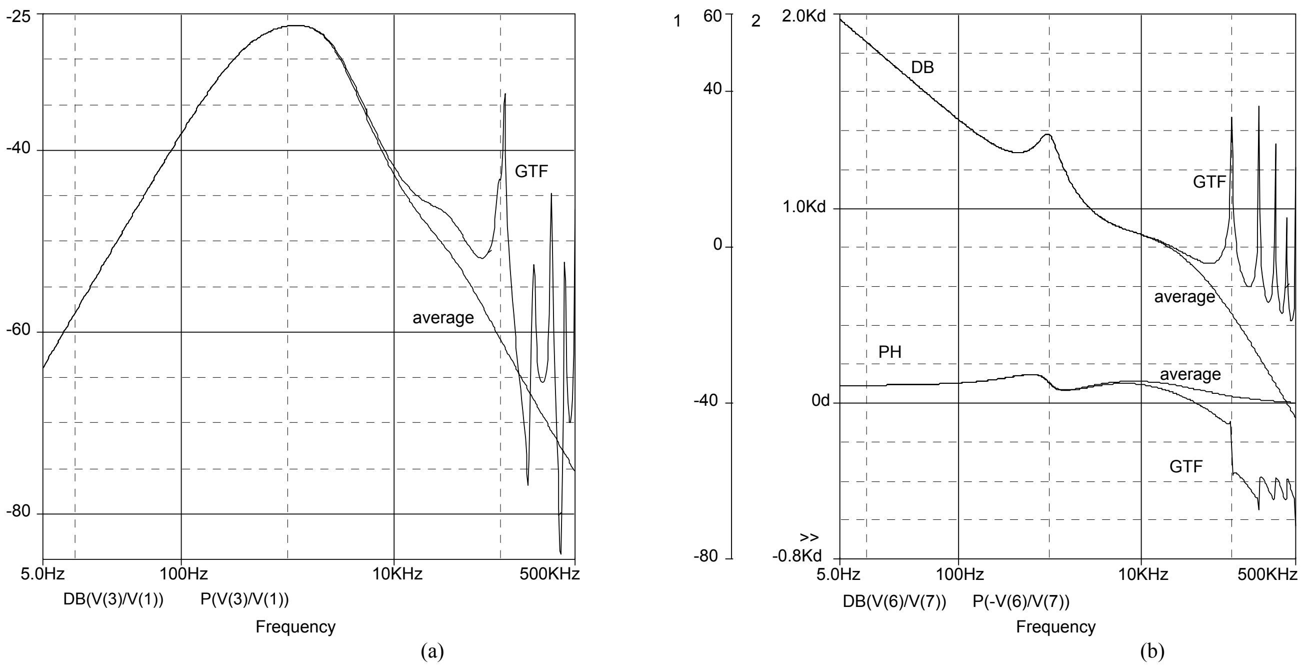

Figure 3. PSPICE simulation of buck regulator in Fig. 2: line-to-output frequency response (a), loop gain (b).

The frequency responses with "averaging" approach do not reflect the switching processes. Their validity is limited to the frequency region below half the switching frequency, i.e. below 50 kHz.

The "GTF" frequency responses are full representatives of converter generalized transfer functions. Then the line-to-output response models in a complex manner the crosstalk of wideband input signal to the output, without the $f_{switch}/2$ limitation. For example, it can be deduced from this response how the individual spectral terms of switching signal will leak into the converter output through poorly shielded input terminals.

In some cases, the "average" and "GTF" comparison can reveal significant differences also in the frequency band below one half of the switching frequency. These differences can be particularly apparent when considering losses inside the power chain, especially the Ron resistances of the switches, and ESRs of the inductance and capacitance. However, this comparative analysis is made difficult due to the convergence problems during the simulation, which are connected with the averaged models from [9].

V. CONCLUSIONS

A novel method of AC analysis of switched DC-DC converters is described. This method utilizes the so-called generalized transfer functions. In comparison with the classical methods based on average modeling, the advantages consist in a more faithful modeling of converter behavior, particularly in the frequency range around $f_{switch}/2$, as well as in the ability to model correctly the transfers above this border frequency. A drawback can be seen in the more complicated mathematical models and thus in their more difficult implementation in current simulation programs.

Computing the control-to-output GTF requires the identification of the DC operating point. Analytical equation (3) can help to find it. Unlike the classical "averaging" modeling, the GTF approach is not accompanied by convergence problems.

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