# GENERALIZED TRANSFER FUNCTIONS OF SWITCHED CIRCUITS

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## Abstract

*Theory of generalized transfer functions (GTF) of networks with periodically time–varying parameters is presented. Focus is concentrated upon the switched circuits. The proposed method is based on the equivalent signal theory by Tsividis. The GTF is a special case of parametric transfer function K(w,t). On the other hand, the well known system functions K(z) of idealized switched networks are special case of the GTF. In many events, the GTF is suitable for the real switched circuits description. In contrast to transfer function K(w,t), its computer implementation is without fundamental difficulties.*

## Introduction

In case of general switched circuits or circuits with periodically time–varying parameters, the response contains both discrete and analog parts. In many cases, the analog part is not fully described by means of z–transformation. The SH circuits and real switched capacitor circuits (SC) are typical cases.

Description of real linear switched networks was solved successfully by many authors. The complex access is described in [1], including the computer–aided algorithm. This algorithm provides – among others – the spectral components of the output signal. These components arise as a consequence of sampled – character signals in switched networks and their computation is numerically expensive.

In some applications, the high – frequency spectral components above the half of sampling frequency are out of our attention. We can use the below described generalized transfer functions (GTF) for these purposes. Analysis is simplified, because the complicated output signal is replaced by so–called "equivalent" smooth signal in accordance with Tsividis [2]. The equivalent and original signals have only some common points interesting for us, and the frequencies of non–zero spectral components of equivalent signal fulfill condition ($f_s$ is the sampling frequency)

$$f < f_s/2 \tag{1}$$

The GTF is defined as the ratio of the Fourier transformation of the equivalent output and input signals.

The way of GTF definition and computation is explained in two simple examples. The computer–aided algorithm will be published separately.

## GENERALIZED TRANSFER FUNCTIONS

**Example 1: SH circuit**

The principal scheme and signals of SH circuit are shown in Fig.1. Following equations are true:

$$v_2(kT+T_1) = v_2(kT)g_1^* + \int_0^{T_1} g(\xi)\, v_1(kT+T_1-\xi)\,d\xi\ , \tag{2}$$

$$v_2(kT+T) = v_2(kT+T_1)g_2^*, \tag{3}$$

where

$$g_1^* = e^{-T_1/\tau_1},\ g_2^* = e^{-T_2/\tau_2},\ g(t) = \frac{1}{\tau_1}e^{-t/\tau_1},\ \tau_1 = \frac{R_s R}{R_s+R}C,\ \tau_2 = RC.$$

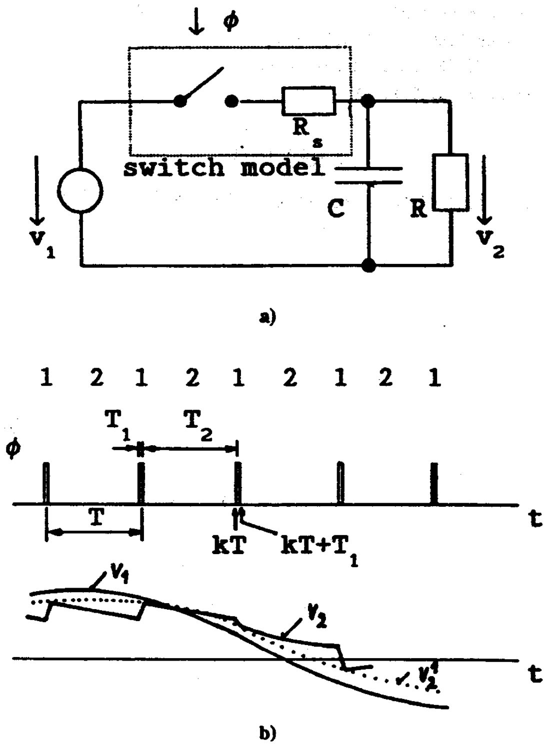


**Figure 1 – a) Principal scheme of SH circuit, b) operating diagram**

Equations (2) and (3) yield the single recurrent formula for the samples $v_2(kT+T_1)$:

$$v_2(kT+T_1) = v_2(kT+T_1-T)g_1^{\bullet}g_2^{\bullet} + \int_0^{T_1} g(\xi)v_1(kT+T_1-\xi)\,d\xi\,. \qquad (4)$$

Peculiarity of this formula consists in the presence of the convolution integral, representing the "analog component" in the circuit behaviour. This formula can be solved by means of the equivalent signals.

Evaluating of the quality of sample selection passes in 3 items:

1 We assume, that condition (1) is true for the input signal $v_1$.

2 We are only interested in the values of output signal in the time instances $(kT+T_1)$, where the sample selection is finished. The signal $v_2$ is replaced by equivalent signal $v_2^1$ (see Fig.1b) with following properties (the subscript 1 denotes the fact, that the equivalent signal corresponds with the samples in the end of switching phases 1):

a) Both signals are identical in the watched instances:

$$v_2^1(kT+T_1) = v_2(kT+T_1)\,,\ k = \pm(0,1,2,\ldots)\,.$$

b) All non-zero spectral components of signal $v_2^1(t)$ fall into the spectral area of input signal $v_1(t)$. The condition (1) is then fulfilled for the signal $v_2^1(t)$.

These conditions result in the knowledge that *the signal is identical with own equivalent signal, if the Nyquist's condition (1) is true.*

3 We compare signals $v_1(t) = v_1^1(t)$ and $v_2^1(t)$. In case of their identity, the samples selection is without mistakes. The comparison in the frequency domain leads to the generalized transfer function $K^{11}(\omega)$:

$$K^{11}(\omega) = \frac{F[v_2^1(t)]}{F[v_1^1(t)]} = \frac{V_2^1(\omega)}{V_1^1(\omega)}\,.$$

The superscripts 11 denote the superscripts of the output and input equivalent signals, respectively. If the condition (1) is true for the input signal, the second superscript can be omit:

$$K^1(\omega) = \frac{V_2^1(\omega)}{V_1(\omega)}\,.$$

Recurrent formula (4) is re-writed for equivalent signals:

$$v_2^1(t) = v_2^1(t-T)g_1^{\bullet}g_2^{\bullet} + \int_0^{T_1} g(\xi)v_1(t-\xi)\,d(\xi)\ , \qquad (5)$$

because the conditions a) and b) are fulfilled:

a) Eq.(5) was obtained from Eq.(4) by condition

$$kT+T_1 = t\,. \qquad (6)$$

b) The signal $v_2^1(t)$ was obtained from the signal $v_1(t)$ by means of linear operations. Then the item b) is fulfilled.

Applying the Fourier transformation to the (5) yields

$$K^1(\omega) = \frac{V_2^1(\omega)}{V_1(\omega)} = G(\omega)\frac{1}{1-e^{-j\omega T}g_1^{\bullet}g_2^{\bullet}}\,, \qquad (7)$$

$$G(\omega) = \int_0^{T_1} g(\xi)\,e^{-j\omega\xi}d\xi = \frac{R}{R+R_s}\,\frac{1-g_1^{\bullet}e^{-j\omega T_1}}{1+j\omega\tau_1}$$

The GTF (7) consists of the part G(w) describing the "analog behaviour", and of the next part with periodical frequency character. This part describes the "discrete behaviour" of SH circuit. Example of GTF for $f_s = 50$ kHz, $T_1 = 2\mu$s, R = 100 kΩ, C = 1 nF and $R_s$ = 1 kΩ, 5 kΩ and 10 kΩ, is in Fig.2.

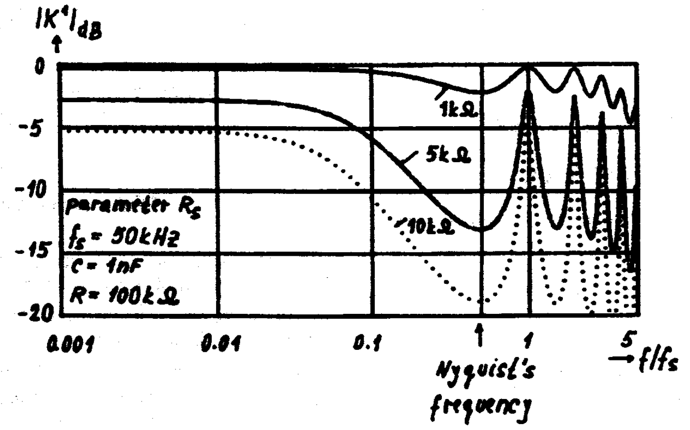


**Figure 2 – Frequency responses of SH circuit**

It is possible to prove that the GTF can also describe the circuit behaviour outside the frequency interval (1). However, the frequency overlapping effect may not occur. We can also investigate the stroboscopic phenomenon and sampling of narrow-band signals.

Let us generalize conditions mentioned above and describe common procedure for the GTF destination:

1 We determine recurrent formula type (4) for the computed network variable in the time instances we are interested.

2 We replace discrete variable by the continuous time (see Eq.6). The equation for equivalent signals arise (see Eq.5).

3 Applying Fourier transformation to this equation, we determine the GTF.

**Example 2: Switched capacitor integrator**

The well known SC integrator scheme is in Fig.3. The switch resistor influence upon the frequency response is investigated. The OpAmp is assumed to be ideal.

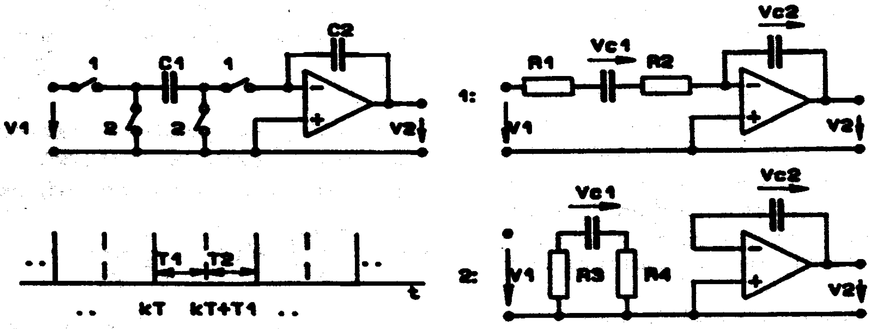


**Figure 3 – SC integrator and its models for phases 1 and 2**

Four equations for two switching phases are true:

$$v_{c1}(kT+T_1) = v_{c1}(kT)\, g_1^* + \int_0^{T_1} g_1(\xi)\, v_1(kT+T_1-\xi)\, d\xi \ ,$$

$$v_{c2}(kT+T_1) = v_{c2}(kT) + \frac{C_1}{C_2}\,[v_{c1}(kT+T_1) - v_{c1}(kT)] \ .$$

$$v_{c1}(kT+T) = v_{c2}(kT+T_1)\, g_2^* \ ,$$

$$v_{c2}(kT+T) = v_{c2}(kT+T_1) \ ,$$

where

$$g_1^* = e^{-T_1/\tau_1} \ , g_2^* = e^{-T_2/\tau_2} \ , g_1(t) = \frac{1}{\tau_1} e^{-t/\tau_1} \ ,$$

$$\tau_1 = (R_1+R_2)C_1 \ , \tau_2 = (R_3+R_4)C_1 \ .$$

Applying the aforementioned procedure for the GTF destination yields:

$$K^1(\omega) = \frac{V_2^1(\omega)}{V_1(\omega)} = -\frac{C_1}{C_2}\,\frac{G^1(\omega)}{1-a\,e^{-j\omega T}} \ ,$$

$$K^2(\omega) = \frac{V_2^2}{V_1(\omega)} = K^1(\omega)\, e^{-j\omega T_2} \ ,$$

where

$$G^1(\omega) = \frac{1-g_1^* e^{-j\omega T_1}}{1+j\omega\tau_1} \ ,$$

$$a = 1 - \frac{C_1}{C_2} g_2^* + \frac{C_1}{C_2} g_1^* g_2^* \ .$$

**Analysis of these results leads to following generalization:**

a) The GTFs are not periodically functions of frequency as in case of discrete systems.

Owing to the inertia of the real switched circuit, the stroboscopic phenomenon is not performed to the full degree, and the transmission fall to zero with the increasing frequency.

b) The GTF becomes periodically function of frequency if the input branch inertia is neglected.

The condition $R_1 + R_2 \rightarrow 0$ yields

$$K^1(\omega) \rightarrow -\frac{C_1}{C_2}\,\frac{1}{1-a\,z^{-1}}\Bigg|_{z=e^{j\omega T}}$$

In this case, we can see that the GTF is described by means of the classical z operator.

c) Neglecting all resistor elements in the switched circuit, the GTFs converge to the classical system functions K(z) of idealized SC circuits:

$R_3 + R_4 \rightarrow 0 \ \Rightarrow$

$$K^1(\omega) \rightarrow -\frac{C_1}{C_2}\,\frac{1}{1-z^{-1}}\Bigg|_{z=e^{j\omega T}} \ ,$$

$$K^2(\omega) \rightarrow -\frac{C_1}{C_2}\,\frac{z^{-T_2/T}}{1-z^{-1}}\Bigg|_{z=e^{j\omega T}} \ .$$

## Wider connections

The GTF represents generalizing of the well-known system functions for ideal SC circuits. On the other hand, the GTF is a special case of parametric transfer functions $K(\omega,t)$ for non-stationary linear systems.

The linear non-stationary system is described by the impulse response $g_*(t,\xi)$ [3]. This is the system response in the time t on the Dirac impulse, that affects on system input at the time $t-\xi$. If the system parameters change synchronously and periodically with the period T, it is true

$$g_*(t,\xi) = g_*(t+iT,\xi),\ i = \pm(0,1,2,\dots) \ .$$

We can compute the system response $v_2(t)$ on the arbitrary input $v_1(t)$:

$$v_2(t) = \int_{-\infty}^{\infty} g_*(t,\xi)\, v_1(t-\xi)\, d\xi \ . \qquad (8)$$

It is possible to derive the relation between the Fourier transforms of signals $v_1(t)$ and $v_2(t)$:

$$V_2(\omega,t) = K(\omega,t)\; V_1(\omega)$$

where

$$K(\omega,t) = \int_{-\infty}^{\infty} g_*(t,\xi)\; e^{-j\omega\xi}\, d\xi \;.$$

is the parametric transfer function of non–stationary system.

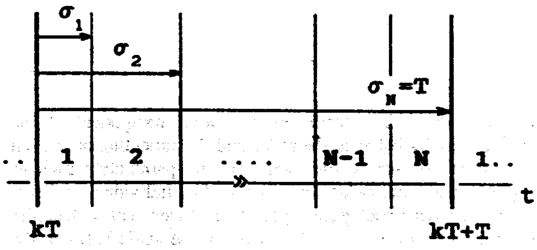


**Figure 4 – Switching diagram of general N–phase circuit**

Let us consider the switched circuit with N–phase switching [1] (see Fig. 4). The equations derived from (8) are true for samples $v_2(kT+\sigma_i)$:

$$v_2(kT+\sigma_i) = \int_{-\infty}^{\infty} g_*(kT+\sigma_i\,,\xi\,)\,v_1(kT+\sigma_i-\xi\,)\cdot d\xi =$$

$$= \int_{-\infty}^{\infty} g_*(\sigma_i\,,\xi)\,v_1(kT+\sigma_i-\xi\,)\; d\xi \;,$$

$$k = \pm(0, 1, 2, \ldots),\; i = 0, 1, \ldots, N-1 \,.$$

The equivalent signal $v_2^i(t)$ corresponds to these samples:

$$v_2^i(t) = \int_{-\infty}^{\infty} g_*(\sigma_i\,,\xi)\,v_1(t-\xi)\,d\xi \;.$$

Applying the Fourier transformation yields the GTF

$$K^i(\omega) = \frac{V_2^i(\omega)}{V_1(\omega)} = \int_{-\infty}^{\infty} g_*(\sigma_i\,,\xi)\,e^{-j\omega\xi}\,d\xi = K(\omega,\sigma_i) \;.$$

*The GTF $K^i(\omega)$ of switched networks is the special case of parametric transfer function for time instances, corresponding to the end of ith switching phases.*

## Conclusions

The aforementioned considerations can be generalized to following conclusions (for more details see [4],[5],[6],[7]):

1 The transmission properties of linear non–stationary systems with periodically varying parameters can be described by means of GTF. The focus is concentrated upon the switched circuits in this paper.

2 GTF is the ratio of Fourier transforms of the equivalent output and input signals. It is possible to set more equivalent signals to the original signal, which does not fulfill Nyquist's condition (1). In this reason, we can describe single circuit by more GTF. We choose the type one in accordance with the time instances, which are essential for us by circuit analysis.

3 The GTF is the special case of parametric transfer function $K(\omega,t)$ for time instances, which were taken account by choosing of equivalent output signal.

4 In general, the GTF of real switched circuits is not periodical function of frequency. The periodic performance is possible in two cases:

a) The separated part of circuit, comprising the input gate, consists only of the reactance elements of the same type, of the ideal transform cells, and of the idealized switches (with $R_{on} = 0$ and $R_{off} \to \infty$). We can consider that these requirements are fulfilled in the number of SC filter LF applications.

b) The input signal remains constant during given switch phases. This assumption is sufficiently fulfilled if the Nyquist's condition is fulfilled with sufficient reserve. In case of the ideal S–H circuit is used for the input samples selection, this assumption is performed exactly.

5 The GTF of switched circuits passes automatically to the well–known system functions K(z) of idealized SC networks, if the real model of circuit is simplified.

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