Topology Transformations for Symbolic Analysis

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***Abstract*—This paper deals with circuit topology transformations for approximate symbolic analysis of linearized circuits. The method is based on two-graph approach. The structure of current and voltage graphs is modified in order to decrease the number of common spanning trees, i.e. the number of symbolic terms.**

I. INTRODUCTION

One of the major problems with the symbolic analysis of linearized electrical circuits is the complexity of the symbolic solutions, which increases exponentially with the number of circuit elements. A symbolic expression is generally valid for all operating conditions as long as the circuit model is valid. In practice, the frequency band of interest is limited and the values of network parameters are limited too. In this case, the majority of symbolic terms can be removed from large expressions without any significant numerical error [2]. This is the basic principle of symbolic simplification methods that have been studied since the late 80s. The simplification process is based on numerical comparison of the original and the simplified solution. It stops when a maximum error is reached.

Our method belongs into the Simplification Before Generation (SBG) class where graphs or matrices representing the circuit are modified (simplified) before the symbolic analysis in order to decrease the number of symbolic terms. A method, published as *Sifting Approach* [2], is based on a heuristic algorithm consisting in device parameter elimination from numerator and denominator submatrices separately. Another method [3], called *Two-graph Simplification*, modifies voltage and current graphs constructed for numerator and denominator separately. The simplification consists in deleting or contracting edges representing a network parameter with sensitivity-based control strategy. The only commercially available symbolic simplification tool – *Analog Insydes* – implements a solution based method [4] that simplifies network matrix by removing its elements.

This paper deals with a different approach. It is based on two-graph method for analysis of linear circuits. Instead of deleting or contracting graph edges it modifies (simplifies) the graph structure in order to decrease the number of common spanning trees. The simplified graphs can be used for subsequent symbolic solution or for a simplified circuit model generation.

Section II briefly introduces the two-graph method, Sections III and IV deal with graph transformations and section V provides a circuit example.

II. THE TWO-GRAPH METHOD

The method of Mayeda and Seshu [5] was originally developed for RLCg_m circuits. Later, it was extended for other active elements including the ideal operational amplifier [6]. The basic idea consists in formulating separately the Kirchhoff's Voltage Law (KVL) and the Kirchhoff's Current Law (KCL) by means of two graphs – the voltage graph G_V and the current graph G_I. Each circuit element (RLCg_m) is represented in each graph by just one edge.

The nodal admittance matrix of a circuit is

$$\mathbf{Y} = \mathbf{A}_I \mathbf{Y}_b \mathbf{A}_V^T , \tag{1}$$

where $\mathbf{A}_I$ and $\mathbf{A}_V$ are the reduced incidence matrices of G_I and G_V, respectively, and $\mathbf{Y}_b$ is the diagonal matrix of circuit parameters. Using the Binet-Cauchy theorem we can write

$$\det \mathbf{Y} = \sum_{t \in T(G_V) \cap T(G_I)} \varepsilon(t) Y^{(t)} , \tag{2}$$

where $Y^{(t)}$ is the tree admittance product of tree t, $T(G_V)$ and $T(G_I)$ are the sets of all spanning trees of voltage and current graphs. The intersection $T(G_V) \cap T(G_I)$ represents common spanning trees of G_I and G_V, i.e. spanning trees that exist in both graphs. $\varepsilon(t) = \pm 1$ is the tree sign. The technique of augmented circuit [6] allows computing both numerator and denominator using (2).

The method possesses the cancellation-free property. If all edges have a unique symbol there are no two identical tree admittance products in (2) that would cancel each other.

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III. TOPOLOGICAL TRANSFORMATIONS

Let G be a graph. Then $V(G)$ is a set of its vertices, $E(G)$ is a set of its edges, and $T(G)$ is a set of its spanning trees (trees for short). Symbol $|E|$ represents the cardinality of set E. The incidence of edge e in graph G, $\rho(e,G) = (i, j)$, assigns two vertices i, j to edge e. A loop is a cyclic sequence of edges and vertices containing each element at most once. An edge with the incidence (v,v) is called selfloop. Graph G is said to be separable if there is a vertex whose removal disconnects the graph into two or more components (a maximal connected subgraph). A block is a maximal nonseparable subgraph [1].

The basic operation of the topological simplification is the separation of a connected subgraph.

Definition 1: *Separation* of a connected subgraph G_S from a graph G is an operation that transforms G into

$$G' = \widetilde{G} \cup G_S \quad , \tag{3}$$

where $\widetilde{G}$ is a subgraph whose edge set is

$$E(\widetilde{G}) = E(G \setminus G_S) \,, \tag{4}$$

where $G \setminus G_S$ is a complement of G_S in G. The incidence $\rho(e,G) = (v_i, v_j)$ of any edge $e \in E(\widetilde{G})$ is transformed by

$$\rho(e,\widetilde{G}) = (f(v_i), f(v_j)) \quad , \tag{5}$$

where f is a mapping of vertex set $V(G \setminus G_S)$ on vertex set $(V(G) \setminus V(G_S)) \cup \{v_c\}$ such that

$$f(v) = \begin{cases} v_c & if \;\; v \in V(G_S) \\ v & otherwise \end{cases} \quad , \tag{6}$$

where v_c is an arbitrary but fixed vertex $v_c \in V(G \setminus G_S) \cap V(G_S)$. The operation will be denoted by symbol $\triangleright$ as

$$G' = G \triangleright G_S \quad . \tag{7}$$

Fig. 1 demonstrates the separation of subgraph $G_S = \{e_1, e_2, e_3\}$ from a graph G. Graph G' was derived from graph G by means of changing the incidence of some edges. Then evidently $|V(G')| = |V(G)|$ and G' is connected. Changing the incidence by (6) may lead to the occurrence of selfloops, but $|E(G')| = |E(G)|$.

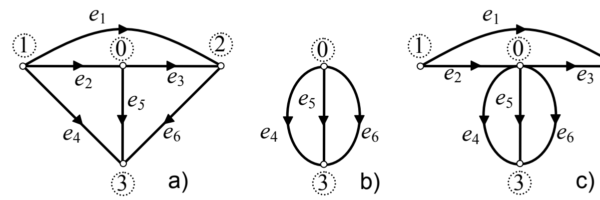

Figure 1. Example of separation $G_S = \{e_1,e_2,e_3\}$ from G: a) original graph G; b) graph $\widetilde{G}$; c) $G' = G \triangleright G_S$.

Lemma 1: Let G and G' be two connected graphs for which $E(G') \subseteq E(G)$ and $|V(G')| = |V(G)|$ holds. If for any loop $L \subseteq G$, $E(L) \subseteq E(G')$ there exists a loop $L' \subseteq G'$ such that $E(L') \subseteq E(L)$, then for any tree $t' \in T(G')$ there exists a tree $t \in T(G)$ such that $E(t') = E(t)$.

Proof. Let us assume that conditions of Lemma 1 hold, yet there exists a tree $t' \in T(G')$ such that $t' \notin T(G)$. Then there must be a subgraph $G_S \subseteq G$ induced by $E(G_S) = E(t')$ containing at least one loop $L \subseteq G_S$ because it is not a tree in G. However, if there exists a loop $L' \subseteq G'$ for L such that $E(L') \subseteq E(L)$, then $E(L') \subseteq E(t')$ because $E(L) \subseteq E(G_S)$ and $E(G_S) = E(t')$. This is in contradiction with the initial assumption that subgraph t' is a tree in G'.

Theorem 1: Let G be a connected nonseparable graph consisting of at least two edges. The separation of any connected subgraph $G_S \subset G$ consisting of at least one edge decreases the number of trees of the transformed graph $G' = G \triangleright G_S$ without the occurrence of alien trees (i.e. trees not presented in the original solution).

Proof. The number of trees is being decreased with the aim of decreasing the number of common trees of voltage and current graphs. The proof strategy consists of verifying: (a) the absence of alien trees, (b) the decreasing the number of trees.

a) The absence of alien trees will be proved by verifying the conditions of Lemma 1. Since $|V(G')| = |V(G)|$ and $E(G') = E(G)$, then it is sufficient to show that for any loop $L \subseteq G$ there exists a loop $L' \subseteq G'$ such that $E(L') \subseteq E(L)$. Let us consider a loop L in graph G. As subgraphs G_S and $G \setminus G_S$ are edge-disjoint and $G = G_S \cup (G \setminus G_S)$, only one of the following three cases may happen:

1. The whole loop L is contained in G_S, i.e. $E(L) \subseteq E(G_S)$. Then there exists a loop $L' \subseteq G'$ such that $E(L') = E(L)$, because G_S is also a subgraph of G'.

2. Loop L is contained both in G_S and $G \setminus G_S$, i.e. $(E(L) \cap E(G_S) \neq \emptyset) \wedge (E(L) \cap E(G \setminus G_S)) \neq \emptyset)$. Let $G_C = L \cap (G \setminus G_S)$, then there exists at least one open trail with endpoints v_i, $v_j \in V(G_S)$ in G_C. Both these endpoints are transformed by (6) into v_C, which forms at least one loop $L' \subseteq G'$.

3. The whole loop L is contained in $G \setminus G_S$, i.e. $E(L) \subseteq E(G \setminus G_S)$. Transformation (6) of $G \setminus G_S$ causes the

occurrence of one or several loops or selfloops L_i' such that $E(L_i') \subseteq E(L)$ holds for each of them.

b) Subgraph $G \setminus G_S$ may be disconnected. Therefore, let us consider an arbitrary component $C \subseteq G \setminus G_S$ and its tree $t_C \in T(C)$. Evidently, there exists a tree $t \in T(G)$ such that $t_C \subset t$. Since G is nonseparable, subgraph C and its complement $G \setminus C$ have at least two vertices u and v in common [1] such that $u, v \in E(G_S)$. Vertices u and v are transformed by (6) into a single vertex. This creates a loop $L' \subset G'$ such that $E(L') \subset E(t)$. Thus the edges of t cannot be a tree in G'. Therefore, the separation decreases the number of trees.

IV. Simplification of Circuit Model

Let the circuit be represented by the current graph G_I and the voltage graph G_V with edges $e_1, e_2, \dots, e_b$, whose weights are magnitudes of (branch) currents $\mathbf{i} = [i_1, i_2, \dots i_b]^T$ and voltages $\mathbf{v} = [v_1, v_2, \dots v_b]^T$ computed for a particular frequency. Any transformation of G_V and G_I affects both numerator and denominator of the network function.

All KVL equations can be written as

$$\mathbf{B}_V \ \mathbf{v} = \mathbf{0} \quad , \tag{8}$$

where $\mathbf{B}_V$ is the complete loop-edge incidence matrix of the voltage graph [6].

Let us assume that (8) consists of B equations with corresponding loops $L_1, L_2, \dots, L_B \subseteq G_V$. The voltage $v(e_j)$ of an edge $e_j \in E(L_i)$ will be considered numerically negligible in loop L_i if

$$\left|v(e_j)\right| < \varepsilon_V \max_{h \in E(L_i)} \left|v(h)\right| \quad , \tag{9}$$

where $\varepsilon_V \in (0, 1)$ is a threshold value. Provided that it causes an acceptable numerical error and does not generate alien trees it is possible to remove all negligible edges from L_i.

Let us assume that the voltage graph G_V can be decomposed into two edge-disjoint subgraphs G_H^V and G_L^V such that

$$G_V = G_H^V \cup G_L^V \tag{10}$$

and the condition

$$\max_{e \in G_L^V \cap L} \left|v(e)\right| < \varepsilon_V \max_{e \in L} \left|v(e)\right| \tag{11}$$

holds for any loop $L \subseteq G_V$ that is contained in both subgraphs. Then it is possible to remove the edges of G_L^V from all such loops. This corresponds to the separation of subgraph G_L^V from graph G_V

$$G'_V = G_V \triangleright G_L^V \ . \tag{12}$$

Generally, subgraph G_L^V may be disconnected with separable components. The separation of a disconnected subgraph is performed component-by-component. Condition (11) has been formulated for loops contained in both subgraphs. Let us assume that G_L^V is separable with blocks B_i^V. As there is no loop contained in two or more blocks, the set of edges satisfying (11) can be divided into disjoint subsets according to B_i^V. Thus, the separation of any block B_i^V, which is considered the *elementary operation*, satisfies (11) as well.

All KCL equations can be formulated for cuts as

$$\mathbf{Q}_I \, \mathbf{i} = \mathbf{0} \quad , \tag{13}$$

where $\mathbf{Q}_I$ is the complete cutset-edge incidence matrix of the current graph [6].

Let us assume that (13) consists of Q equations corresponding to cuts $C_1, C_2, \dots, C_Q \subseteq G_I$. The current $i(e_j)$ of a cut edge $e_j \in E(C_i)$ will be considered numerically negligible in C_i if

$$\left|i(e_j)\right| < \varepsilon_I \max_{h \in E(C_i)} \left|i(h)\right| \quad , \tag{14}$$

where $\varepsilon_I \in (0, 1)$ is a threshold value. Provided that it causes an acceptable numerical error and does not generate alien trees it is possible to remove such edges from C_i.

It is convenient to reformulate criterion (14) for cut edges into another criterion for loops of the current graph. Therefore, let us consider the current graph G_I and its arbitrary loop $L \subseteq G_I$ containing an edge $e_{min} \subseteq L$ such that

$$|\, i(e_{min})\,| = \min_{e \in E(L)} |\, i(e)\,| \ . \tag{15}$$

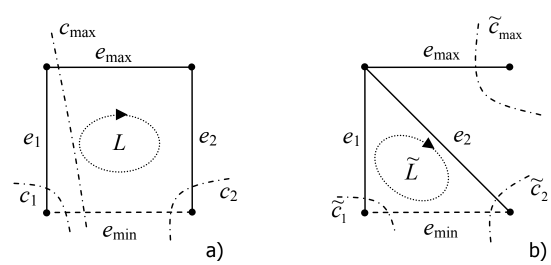

Figure 2. a) Loop in G_I ; b) Transformed graph.

Let t be a tree of G_I such that $L \setminus \{e_{min}\} \subseteq t$. Except for e_{min}, the edges of loop L are elements of the tree as illustrated in Fig. 2a. Let graph G_I be decomposed into two edge-disjoint subgraphs

$$G_{\mathrm{I}} = G_{\mathrm{H}}^{\mathrm{I}} \cup G_{\mathrm{L}}^{\mathrm{I}} \ , \tag{16}$$

where $G_{\mathrm{H}}^{\mathrm{I}}$ contains all edges e of G_I for which

$$|\, i(e_{\min}) \,| < \varepsilon_{\mathrm{I}} \,|\, i(e) \,| \ . \tag{17}$$

Subgraph $G_{\mathrm{L}}^{\mathrm{I}}$ contains the remaining edges.

In accordance with (14) it is possible in all basic cuts defined by edges $t \cap G_{\mathrm{H}}^{\mathrm{I}}$ and containing e_{min} to neglect $i(e_{min})$. For appropriate edge numbering the basic cutset matrix [7] of transformed graph $\widetilde{G}_{\mathrm{I}}$ will be

$$\tilde{\mathbf{Q}} = \left[\begin{array}{ccc|c:c} & \overset{e_{\min}}{\mathbf{0}} & & \overbrace{\mathbf{E}_{\mathrm{H}}}^{t \cap G_{\mathrm{H}}^{\mathrm{I}}} & \overbrace{\phantom{\mathbf{E}_{\mathrm{L}}}}^{t \cap G_{\mathrm{L}}^{\mathrm{I}}} \\ \hdashline & & & & \mathbf{E}_{\mathrm{L}} \end{array}\right] , \tag{18}$$

where $\mathbf{E}_H$ is a unity matrix corresponding to the edges of $t \cap G_{\mathrm{H}}^{\mathrm{I}}$, and $\mathbf{E}_L$ is a unity matrix corresponding to the edges of $t \cap G_{\mathrm{L}}^{\mathrm{I}}$.

The major submatrix $\mathrm{diag}(\mathbf{E}_{\mathrm{H}}, \mathbf{E}_{\mathrm{L}})$ of tree t is preserved by the $G_{\mathrm{I}} \to \widetilde{G}_{\mathrm{I}}$ transformation and therefore edges corresponding to the major form a tree $\tilde{t}$ also in $\widetilde{G}_{\mathrm{I}}$ [1]. The basic loopset matrix with respect to tree $\tilde{t}$ can be obtained directly from $\tilde{\mathbf{Q}}$ [7] as

$$\tilde{\mathbf{B}} = \left[\begin{array}{c|c:c} & \overbrace{\phantom{\mathbf{0}}}^{t \cap G_{\mathrm{H}}^{\mathrm{I}}} & \overbrace{\phantom{\mathbf{0}}}^{t \cap G_{\mathrm{L}}^{\mathrm{I}}} \\ \mathbf{E} & \mathbf{0} & \\ & & \end{array}\right] \begin{array}{l} \\ e_{\min} \\ \\ \end{array} . \tag{19}$$

The zero submatrix in $\tilde{\mathbf{B}}$ shows that all edges of $G_{\mathrm{H}}^{\mathrm{I}}$ have been removed from the basic loop defined by the tree t and edge e_{min}. Thus all edges e for which

$$\min_{h \in E(L)} |\, i(h) \,| < \varepsilon_{\mathrm{I}} \,|\, i(e) \,| \tag{20}$$

can be removed from any loop L of the current graph if it causes an acceptable numerical error and does not generate alien trees.

The transformation of cut-edge criterion (14) into loop criterion (20) is illustrated in Fig. 2. Let us suppose that edge e_{max} is the only one that fulfils (20) for certain ε_I. Edge e_{max} can be removed from loop L. It is noticeable that the basic cut defined by tree edge e_{max} does not contain edge e_{min} in the modified graph.

Let G_I be decomposed with respect to a threshold value $\varepsilon_I \in (0, 1)$ into two edge disjoint subgraphs $G_{\mathrm{H}}^{\mathrm{I}}$ and $G_{\mathrm{L}}^{\mathrm{I}}$ and for any loop $L \subseteq G_I$ contained in both subgraphs

$$\min_{e \in E(L)} |i(e)| < \varepsilon_{\mathrm{I}} \min_{e \in E(G_{\mathrm{H}}^{\mathrm{I}} \cap L)} |i(e)| \tag{21}$$

holds. Then it is possible in accordance with (20) to remove all edges of $G_{\mathrm{H}}^{\mathrm{I}}$ from any loop contained in both $G_{\mathrm{H}}^{\mathrm{I}}$ and $G_{\mathrm{L}}^{\mathrm{I}}$ by means of the separation

$$G'_{\mathrm{I}} = G_{\mathrm{I}} \triangleright G_{\mathrm{H}}^{\mathrm{I}} \ . \tag{22}$$

Spanning trees of G_V and G_I are reduced separately. It is likely that the reduction in one graph causes also the reduction of common trees.

V. EXAMPLE

The method of topological transformation has been implemented experimentally in MATLAB. First, the circuit is preprocessed by a standard parametric SBG method that removes all negligible network elements by setting their parameters to zero or infinity.

perform parametric simplification preprocessing;
compute reference numerical solution;
while $\varepsilon_A < \varepsilon_{max}$ {
  identify possible graph transformations for certain ε_I *and* ε_V*;*
  compute error of each transformation;
  perform operation with the lowest error;
  update accumulated error ε_A;
  update numerical solution;
}
undo last transformation;

Figure 3. Main cycle of the method.

The error-control strategy of our method is similar to the method of [4]. Large-change sensitivity is used for the transformation ranking. Transformation with the lowest influence to the network function is performed and the ranking is updated. The procedure stops when the maximum approximation error is reached.

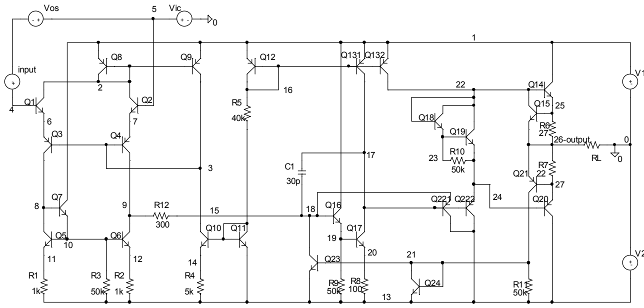

Figure 4. Circuit diagram of μA741 operational amplifier [8].

The topological transformation is demonstrated on an analysis of the μA741 operational amplifier. Its circuit diagram has been borrowed from [8], Fig. 4. The aim was to simplify the circuit model for open-loop gain in the neighborhood of the dominant pole frequency of 3 Hz. The required accuracy was 1.5 dB for magnitude and 3° for phase checked at frequencies of 0.1 Hz and 5 Hz.

The parametric preprocessing reduced 179 network parameters from original 196 to 17. The topological algorithm separated two subgraphs from the voltage graph and two subgraphs from the current graph. A comparison of the graphs before and after simplification in Fig. 5 illustrates the basic principle of the method. To keep the drawing neat, individual blocks are drawn separately. The graphs are, of course, connected. The increasing of graph blocks automatically decreases the number of common spanning trees. The number of common spanning trees was decreased from 54 to 2 for the numerator and from 490 to 42 for the denominator.

Conclusions

The method of topological simplification allows reaching a higher degree of simplification than parameter-based SBG techniques do. Its principle consists in such transformations of the voltage and current graphs that decrease the number of common spanning trees. The occurrence of alien trees is avoided by the operations.

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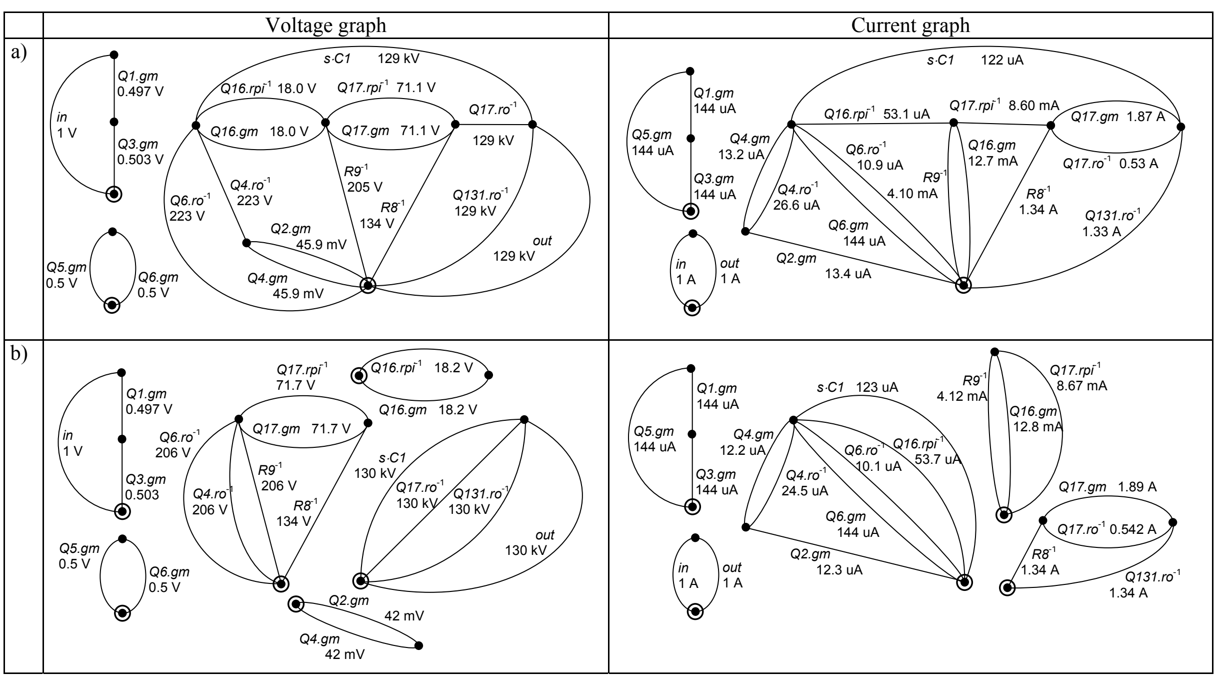

Figure 5. Comparison of voltage and current graphs (a) before and (b) after topological simplification. The numbers at each edge are modules of voltages or currents at 5 Hz. Symbol ⊙ denotes the same vertex in each graph.