Algebraic Propositions for Analysis of Linear-Phase FIR Filters

Viera Biolkova and Zdenek Kolka
Dept. of Radioelectronics
FEEC, Brno Univ. of Technology
Brno, Czech Republic
biolkova@feec.vutbr.cz

Dalibor Biolek
Dept. of EE & Dept. of Microelectronics
FEEC, Brno Univ. of Technology, & FVT, Univ. of Defence
Brno, Czech Republic
dalibor.biolek@unob.cz

***Abstract*—Several algebraic propositions from the theory of reciprocal equations are conceived in this contribution. Applying them to the *z*-domain system functions of discrete-time FIR filters with the symmetric and antisymmetric impulse responses yields basic conditions and rules that have to be fulfilled to reach in principle the linear phase of filters with prescribed amplitude frequency responses.**

I. INTRODUCTION

The algebraic equation

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0\,, \tag{1}$$

whose coefficients a_k (k = 0,1,2, …, n) are in relation (2)

$$a_k = a_{n-k}, \text{ or } a_k = - a_{n-k}\,, \tag{2}$$

is called positively or negatively reciprocal equation [1], [2]. Let us denote it in brief as $R(x) = 0$ or $r(x) = 0$.

Let us consider the z-domain system function of a discrete-time n^{th} -order FIR filter

$$K(z) = \sum_{k=0}^{n} a_k z^{-k} = \frac{\sum_{k=0}^{n} a_k z^{n-k}}{z^n}\,. \tag{3}$$

The well-known conditions of linear-phase FIR filters are so-called symmetrical or antisymmetrical property of impulse response [3]. These conditions are stated by (2). Let us briefly call such a filter with properties (2) symmetrical or antisymmetrical filter.

System function (3) of n^{th} -order FIR filter has n zeros and one n-multiple pole in zero. Finding zeros of such a filter can be transferred into finding the roots of reciprocal algebraic equation. One can find several interesting propositions about roots of such equations in algebra. Their methodical application reveals some conditions which must be fulfilled for low-pass, high-pass, band-pass and band-reject FIR filters to have exactly linear phase responses.

II. ALGEBRAIC PROPOSITIONS

The following three Theorems are given in [1]:

Theorem 1:

Every positively reciprocal equation of odd degree and every negatively reciprocal equation of even degree has the root -1.

Theorem 2:

Every negatively reciprocal equation has the root +1.

Theorem 3 (being short):

Reducing a reciprocal equation by linear factors of the form x-1, x+1, *we get a positively reciprocal equation of even degree.*

An additional Theorem can be found in [2]:

Theorem 4:

Any reciprocal equation has together with a root x also a root 1/*x*.

Starting from these fundamental Theorems, let us formulate the following Propositions:

Proposition 1:

Multiplying any positively (negatively) reciprocal equation of degree n by a root factor (x+1) results in a positively (negatively) reciprocal equation of degree n+1.

Proposition 2:

Multiplying any positively (negatively) reciprocal equation of degree n by a root factor (x-1) results in a negatively (positively) reciprocal equation of degree n+1.

Proposition 3:

Multiplying any positively (negatively) reciprocal equation of degree n by a root factors (x-a)(x-1/a) results in a positively (negatively) reciprocal equation of degree n+2. The coefficients of the resulting

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equation will be real numbers on the assumption that a is either a real number or a complex number with absolute value equal to one.

Proposition 4:

Multiplying any positively (negatively) reciprocal equation of degree n by a root factors (x-a)(x-1/a)(x-a)(x-1/a*) results in a positively (negatively) reciprocal equation of degree n+4. The coefficients of the resulting equation will be real numbers. Here a is a general complex number and a* is the corresponding complex conjugate.*

III. CONSEQUENCES OF ALGEBRAIC THEOREMS AND PROPOSITIONS FOR FIR FILTERS

First recall the relations between the coefficients (or zeros) of the system function $K(z)$ (3) of FIR filter and its frequency response $K(\omega)$:

DC gain is acquired by substituting $z = 1$ into the system function:

$$K_0 = K(\omega)\Big|_{\omega=0} = K(z)\Big|_{z=1} \tag{4}$$

The gain at half the sampling frequency $\omega_s/2$ is obtained by substituting $z = -1$ into the system function:

$$K_{1/2} = K(\omega)\Big|_{\omega=\omega_s/2} = K(z)\Big|_{z=-1} \tag{5}$$

The gain at general frequency ω_0 is computed according to the procedure

$$K_{\omega_0} = K(\omega)\Big|_{\omega=\omega_0} = K(z)\Big|_{z=e^{j\omega_0 T}}, T = \frac{2\pi}{\omega_s} \tag{6}$$

Let the filter for which $K_0 = 0$ ($K_0 \neq 0$) be named AC (DC) filter.

Let the filter for which $K_{1/2} = 0$ be named reject filter or filter with transfer zero at half the sampling frequency.

Let the filter for which $K_{\omega_0} = 0$ for $0 \leq \omega_0 \leq \omega_s/2$ be named filter with transfer zero (at frequency ω_0).

Theorems 1 to 3 and Propositions 1 to 4 lead to the following Implications.

Implication 1:

Any antisymmetrical filter has zero point $z = 1$, *thus it is an AC filter. For the order of* $n > 1$, *this zero point can be separated. The result of this separation is a symmetrical* $(n-1)$*-order filter.*

Implication 2:

Any symmetrical filter of odd order $n = 2k+1$ *has zero point* $z = -1$, *thus it is a reject filter. For the order of* $n > 1$, *this zero point can be separated. The result of this separation is a symmetrical* $2k$ *- order filter.*

Implication 3:

Any symmetrical or antisymmetrical filter, having zero point $z \neq \pm 1$, *has simultaneously the zero point* $1/z$, *as well as the corresponding complex conjugate zero points.*

Implication 4:

Augmenting any symmetrical (antisymmetrical) n-th order filter by arbitrary r zero points $z = -1$ *always yields a symmetrical (antisymmetrical) n+r-order reject filter.*

Implication 5:

Augmenting any symmetrical (antisymmetrical) n-th order filter by arbitrary r zero points $z = 1$ *always yields an n+r-order DC filter. When the number of additional zeros is even, the resulting filter will be symmetrical (antisymmetrical), otherwise it will be antisymmetrical (symmetrical).*

Implication 6:

Augmenting any symmetrical (antisymmetrical) n-th order filter by a couple of zero points $\exp(j\omega_0 T)$, $\exp(-j\omega_0 T)$ *yields an* $(n+2)$*-order symmetrical (antisymmetrical) filter with transfer zero at frequency* ω_0.

Implication 7:

Augmenting any symmetrical (antisymmetrical) n-th order filter by quadruplets of zeros a+jb, a-jb, 1/(a+jb), 1/(a-jb) yields an $(n+4)$*-order symmetrical (antisymmetrical) filter.*

Some of the above conclusions are summarized in Table 1. It should be noted that by means of the symmetrical filter one can implement all basic types of frequency responses. However, the antisymmetrical filter can realize only reject bandpass filters (for both even and odd orders) and reject lowpass filters (only for odd orders).

IV. EXAMPLE

Let us consider an antisymmetrical 10-th order reject AC filter with the following coefficients of impulse response:

TABLE I. A SUMMARY OF THE POSSIBILITIES OF IMPLEMENTING VARIOUS TYPES OF LINEAR-PHASE FIR FILTERS BY SYMMETRICAL AND ANTISYMMETRICAL STRUCTURES.

<table>
<tr><th rowspan="2">filter type</th><th>symmetrical filter</th><th colspan="2">antisymmetrical filter</th></tr>
<tr><th>all orders</th><th>even order</th><th>odd order</th></tr>
<tr><td>low-pass</td><td rowspan="6">yes</td><td rowspan="4">no</td><td>no</td></tr>
<tr><td>low-pass reject</td><td>yes</td></tr>
<tr><td>high-pass</td><td rowspan="2">no</td></tr>
<tr><td>band-pass</td></tr>
<tr><td>band-pass reject</td><td>yes</td><td>yes</td></tr>
<tr><td>band-reject</td><td>no</td><td>no</td></tr>
</table>

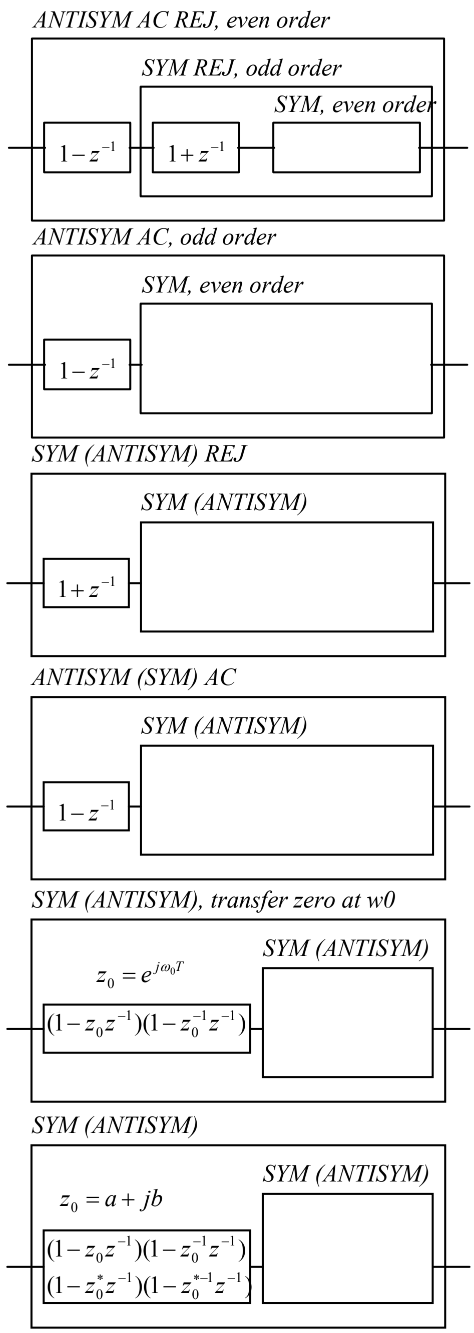

Figure 1. Illustration of Implications 1 to 7.

$[h_0, h_1, ..., h_{10}] =$ [7.20396e-3, 2.19314e-2, 5.29020e-2, 7.93358e-2, 5.88771e-2, 0, -5.88771e-2, -7.93358e-2, -5.29020e-2, -2.19314e-2, -7.20396e-3].

As an Implication 1, this filter has a zero z = 1. Its separation yields the following coefficients of a symmetrical 9-th order DC reject filter:

$[h'_0, h'_1, ..., h'_{10}] =$ [7.20396e-3, 2.91353e-2, 8.20373e-2, 1.61373e-1, 2.2025e-1, 2.20250e-1, 1.61373e-1, 8.20373e-2, 2.91353e-2, 7.20396e-3].

As an Implication 2, any symmetrical filter of odd order has a zero z = -1. Its separation yields the following coefficients of a symmetrical 8-th order DC filter:

$[h''_0, h''_1, ..., h''_{10}] =$ [1.44079e-2, 4.38628e-2, 1.20212e-1, 2.02534e-1, 2.37966e-1, 2.02534e-1, 1.2021e-1, 4.38628e-2, 1.44079e-2].

That is why the original filter can be realized as a cascade connection of first-order and 9-th order FIR filters or as a connection of first-order, first-order, and 8-th order filters:

$$K(z) = \sum_{k=0}^{10} h_k z^{-k} = (1-z^{-1}) \sum_{k=0}^{9} h'_k z^{-k} = (1-z^{-1})(1+z^{-1}) \sum_{k=0}^{8} h''_k z^{-k}$$

All of the sub-sections exhibit linear phase responses. Amplitude frequency responses of the 10-th, 9-th and 8-th sections are compared in Fig. 2

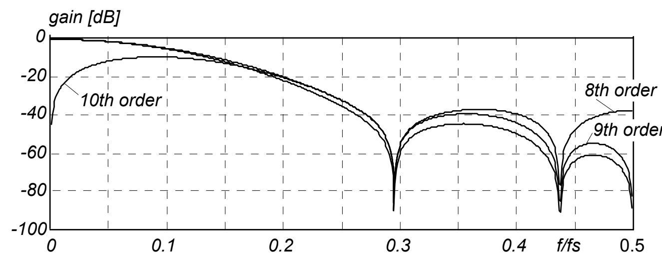

Figure 2. Amplitude frequency responses of filters from example.

V. Conclusion

The paper offers a systematic investigation of mutual relations between the coefficients of FIR filters with linear phase responses and specific types of their amplitude frequency responses. This investigation is based on general algebraic propositions about the so-called positively and negatively reciprocal equations. To enable this analysis, we sort all possible linear-phase FIR filters into newly defined groups of symmetrical and antisymmetrical filters, AC, DC, and reject filters, and filters with transfer zeros at concrete frequencies. It is shown that, in principle, some of the above types of FIR filters must have specific zeros, e.g. z=1 or z=-1, and that this knowledge can help us to decompose FIR filter into specific subblocks. It is also shown that antisymmetrical filters can provide only some selected types of amplitude frequency responses.

Acknowledgment

This work has been supported by the research programmes MSM0021630503 and MSM0021630513.

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