LDI MATRIX AND ITS UTILIZATION FOR THE CIRCUIT ANALYSIS AND DESIGN

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Abstract: *The LDI matrix is introduced in this contribution. Using this matrix, we can recalculate coefficients of the transfer functions of continuous- and discrete-time linear circuits which are connected by the LDI transformation.*

Introduction

The LDI (Lossless Discrete Integration) transformation belongs to the so-called *s-z* transformations which are frequently used mainly for the synthesis and modeling of switched capacitor and switched current circuits. Applying the *s-z* transformation to an *s*-domain transfer function of continuous-time (CT) circuit yields a corresponding *z*-domain transfer function of discrete-time (DT) circuit. If an inverse transformation exists, it is possible to convert the discrete-time transfer function back to the original *s*-domain transfer function.

For such transformations between the continuous- and discrete-time systems which are connected by the concrete *s-z* transformation, the mutual recalculation of the transfer function coefficients is necessary. This recalculation can be done by an effective algorithm which is advantageous especially for the design of high-order systems. In case of the often used bilinear transformation, the algorithm has been referred to as the Pascal matrix [1]. In [2], this algorithm is also generalized for the BD (Backward-Difference) and FD (Forward-Difference) transformation, and for the general first-order *s-z* transformation.

The algorithm for LDI transformation is derived below. The procedure given could also be modified for other parametric transformations which combine the LDI and some first-order transformations.

LDI transformation

LDI transformation is defined as follows:

$$s = \frac{1}{T}\left(z^{1/2} - z^{-1/2}\right), \text{ or } z = 1 + \frac{(sT)^2}{2} \pm \frac{sT}{2}\sqrt{4 + (sT)^2}\ , \tag{1}$$

where T is the sampling period used for a discrete-time circuit.

The exponents $\pm 1/2$ in (1) have their physical interpretation in the switched circuits, which consists in dividing the sampling period into two switching phases. The substitutions

$$z = e^{j\omega T}, \quad z^{\pm 1/2} = e^{\pm j\omega T/2}$$

show that the discrete-time circuit which is designed using the LDI transformation can be considered a circuit with half sampling period (double sampling frequency). As shown below, the *N+1* coefficients of CT circuit are transformed into *2N+1* coefficients of DT circuit.

To simplify further considerations, let us introduce the substitutions

$$x = z^{1/2}, \tag{2}$$

$$a = \frac{1}{T}. \tag{3}$$

Then the LDI transformation can be rewritten:

$$s = a\left(x - \frac{1}{x}\right) = a\frac{x^2 - 1}{x}, \quad x \neq 0\ . \tag{4}$$

Example of the utilization of LDI transformation

Consider a CT second-order bandpass filter with the resonant frequency 1kHz and the quality factor 100:

$$K_{CT}(s)=\frac{b_1 s}{s^2+b_1 s+b_0}=\frac{62{,}8319\mathrm{s}}{s^2+62{,}8319\mathrm{s}+3{,}94784.10^7}\,. \tag{5}$$

This filter could be CT prototype of DT circuit with the sampling frequency 100 kHz and the corresponding sampling period 10μs. According to (3)

$$a = 10^5 \text{ Hz}.$$

We substitute (4) into (5). After the arrangement we obtain

$$K_{DT}(x)=\frac{b_1 a\dfrac{x^2-1}{x}}{\left[a\dfrac{x^2-1}{x}\right]^2+b_1 a\dfrac{x^2-1}{x}+b_0}=\frac{b_1}{a}\,\frac{x\left(x^2-1\right)}{x^4+\dfrac{b_1}{a}x^3+\left(\dfrac{b_0}{a^2}-2\right)x^2-\dfrac{b_1}{a}x+1}=$$

$$=6{,}28319.10^{-4}\,\frac{x\left(x^2-1\right)}{x^4+6{,}28319.10^{-4}x^3-1{,}99605x^2-6{,}28319.10^{-4}x+1}\,.$$

Coming back to substitution (2) yields the z-domain transfer function of DT bandpass filter:

$$K_{DT}(z)=6{,}28319.10^{-4}\,\frac{z^{1/2}(z-1)}{z^2+6{,}28319.10^{-4}z^{3/2}-1{,}99605z-6{,}28319.10^{-4}z^{1/2}+1}\,. \tag{6}$$

The aforementioned facts lead to the conclusion that the given filter can be described as a 4-th order DT circuit with a sampling frequency of 200 kHz.

In the following, we state an algorithm for finding the z (or x) domain coefficients for the general transfer function.

Algorithm

Consider an N-th order CT linear circuit with the transfer function

$$K_{CT}(s)=\sum_{k=0}^{N}a_k s^k\Big/\sum_{k=0}^{N}b_k s^k\,. \tag{7}$$

We apply LDI transformation (4):

$$K_{DT}(x)=K_{CT}\left(s=a\frac{x^2-1}{x}\right)=\frac{\sum\limits_{k=0}^{N}a_k\left(a\dfrac{x^2-1}{x}\right)^k}{\sum\limits_{k=0}^{N}b_k\left(a\dfrac{x^2-1}{x}\right)^k}=\frac{\sum\limits_{k=0}^{N}a_k a^k\left(x^2-1\right)^k x^{N-k}}{\sum\limits_{k=0}^{N}b_k a^k\left(x^2-1\right)^k x^{N-k}}\,. \tag{8}$$

After the power expansion and some algebraic arrangements, we get the general structure of the transformed transfer function:

$$K_{DT}(x)=\sum_{k=0}^{2N}c_k x^k\Big/\sum_{k=0}^{2N}d_k x^k\,. \tag{9}$$

The numerator (denominator) coefficients c_k (d_k) of the DT circuit can be derived from the set of the numerator (denominator) coefficients a_k (b_k) of the CT circuit. Because of the formal identity of the numerator and denominator in (8), it is sufficient to derive transform relations only for the numerator.

Using the binomical rule and making the necessary simplifications, we obtain the following equations:
$0 \le k \le N$:

$$c_k = \sum_{i=0}^{[k/2]} (-1)^{N-k+i} \binom{N-k+2i}{N-k+i} a^{N-k+2i} a_{N-k+2i} , \tag{10}$$

$2N \ge k > N$:

$$c_k = \sum_{i=0}^{N-\left[\frac{k+1}{2}\right]} (-1)^i \binom{k-N+2i}{i} a^{k-N+2i} a_{k-N+2i} , \tag{11}$$

where [ ] is the integer of the argument.

Equations (10) and (11) represent the algorithm how to compute the numerator coefficients of DT circuits from the numerator coefficients of CT circuits. For better transparency, let us set the coefficients a_k of the CT circuit and the coefficients c_k of DT circuit to the vectors

$$\mathbf{A} = [a_0\ a_1\ a_2 \cdots a_N]^T, \quad \mathbf{C} = [c_0\ c_1\ c_2 \cdots c_{2N}]^T . \tag{12}$$

Now we define auxiliary vector

$$\widetilde{\mathbf{A}} = [a_0 a^0\ a_1 a^1\ a_2 a^2 \cdots a_N a^N]^T . \tag{13}$$

Then equations (10) and (11) can be rewritten in the matrix form

$$\mathbf{C} = \mathbf{L}\widetilde{\mathbf{A}} . \tag{14}$$

L is the ($2N+1$) x ($N+1$) matrix whose elements are given by equations (10) and (11). Let us call it the *LDI matrix*.

The rules how to compile an *LDI* matrix can be investigated, for example, from equation (16), for $N = 5$. The matrix element in the upper right corner is always

$$(-1)^N \binom{N}{N} = (-1)^N .$$

The sign of nonzero elements in each row and column alternates.

The following facts result from the structure of *LDI* matrix:

1. Coefficients c_k of DT circuit with even (odd) index k are only determined by the coefficients a_i of CT circuit with odd (even) indexes.
2. Coefficients c_k of DT circuit for $k>N$ are connected with the coefficients for $k<N$ using the relations

$$c_k = (-1)^{N-k} c_{2N-k}, \quad k \le N . \tag{15}$$

It should be noted that the inverse LDI transformation could only be performed under condition (15).

3. The inverse LDI transformation is easy due to the simple structure of the *LDI* matrix.

Example of the utilization of the algorithm

Let us apply our algorithm to the evaluation of the DT bandpass filter from CT prototype (5). For $N = 2$, $a = 10^5$ Hz we get equations (17) and (18):

$$
\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ \hline c_6 \\ c_7 \\ c_8 \\ c_9 \\ c_{10} \end{bmatrix}
=
\left[\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & -\binom{5}{5} \\
0 & 0 & 0 & 0 & \binom{4}{4} & 0 \\
0 & 0 & 0 & -\binom{3}{3} & 0 & \binom{5}{4} \\
0 & 0 & \binom{2}{2} & 0 & -\binom{4}{3} & 0 \\
0 & -\binom{1}{1} & 0 & \binom{3}{2} & 0 & -\binom{5}{3} \\
\binom{0}{0} & 0 & -\binom{2}{1} & 0 & \binom{4}{2} & 0 \\
\hline
0 & \binom{1}{0} & 0 & -\binom{3}{1} & 0 & \binom{5}{2} \\
0 & 0 & \binom{2}{0} & 0 & -\binom{4}{1} & 0 \\
0 & 0 & 0 & \binom{3}{0} & 0 & -\binom{5}{1} \\
0 & 0 & 0 & 0 & \binom{4}{0} & 0 \\
0 & 0 & 0 & 0 & 0 & \binom{5}{0}
\end{array}\right]
\begin{bmatrix} a_0 \\ a_1 a \\ a_2 a^2 \\ a_3 a^3 \\ a_4 a^4 \\ a_5 a^5 \end{bmatrix}
=
\left[\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & -1 & 0 & 5 \\
0 & 0 & 1 & 0 & -4 & 0 \\
0 & -1 & 0 & 3 & 0 & -10 \\
1 & 0 & -2 & 0 & 6 & 0 \\
\hline
0 & 1 & 0 & -3 & 0 & 10 \\
0 & 0 & 1 & 0 & -4 & 0 \\
0 & 0 & 0 & 1 & 0 & -5 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\begin{bmatrix} a_0 \\ a_1 a \\ a_2 a^2 \\ a_3 a^3 \\ a_4 a^4 \\ a_5 a^5 \end{bmatrix}
\tag{16}
$$

The numerator:

$$
\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \hline c_3 \\ c_4 \end{bmatrix}
=
\left[\begin{array}{ccc} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -2 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]
\begin{bmatrix} a_0 \\ a_1 a \\ a_2 a^2 \end{bmatrix}
=
\left[\begin{array}{ccc} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -2 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]
\begin{bmatrix} 0 \\ 62{,}8319.10^5 \\ 0 \end{bmatrix}
=
\begin{bmatrix} 0 \\ -62{,}8319.10^5 \\ 0 \\ 62{,}8319.10^5 \\ 0 \end{bmatrix}
\tag{17}
$$

The denominator:

$$
\begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \hline d_3 \\ d_4 \end{bmatrix}
=
\left[\begin{array}{ccc} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -2 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]
\begin{bmatrix} b_0 \\ b_1 a \\ b_2 a^2 \end{bmatrix}
=
\left[\begin{array}{ccc} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -2 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]
\begin{bmatrix} 3{,}94784.10^7 \\ 62{,}8319.10^5 \\ 10^{10} \end{bmatrix}
=
\begin{bmatrix} 10^{10} \\ -62{,}8319.10^5 \\ -1{,}99605.10^{10} \\ 62{,}8319.10^5 \\ 10^{10} \end{bmatrix}
\tag{18}
$$

Conclusion

Equations (12), (13) and (16) offer directions how to compile a transform matrix of LDI *s-z* transformation which is defined by equation (1) or (4). Equation (14) shows how to use it for recalculating coefficients of CT and DT systems.

Acknowledgement: *This work is supported by the Grant Agency of the Czech Republic under grant No. 102/98/P204.*

References

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